7.1 Introduction to Discrete Probability

Credit

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Terminology

Experiment

- A repeatable procedure that yields one of a given set of outcomes
  - Rolling a die, for example

Sample space

- The set of possible outcomes
  - For a die, that would be values 1 to 6

Event

- A subset of the sample experiment
  - If you rolled a 4 on the die, the event is the 4

Probability

Experiment: We roll a single die, what are the possible outcomes? {1, 2, 3, 4, 5, 6}

The set of possible outcomes is called the **sample space.**

We roll a pair of dice, what is the sample space?

Depends on what we're going to ask.

Often convenient to choose a sample space of equally likely outcomes.

{(1, 1), (1, 2), (1, 3), … , (2, 1), …, (6, 6)}

Probability definition: Equally Likely Outcomes

The probability of an event occurring (assuming equally likely outcomes) is:

$$p(E) = \frac{|E|}{|S|}$$

- Where E is an event corresponds to a subset of outcomes.
  Note: $E \subseteq S$. $|E|$ is the cardinality of E. $|S|$ is the cardinality of S.
- Where S is a finite sample space of equally likely outcomes
- Note that $0 \leq |E| \leq |S|$
  - Thus, the probability will always be between 0 and 1
  - An event that will never happen has probability 0
  - An event that will always happen has probability 1

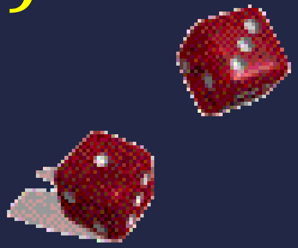
Probability is always a value between 0 and 1

- Something with a probability of 0 will never occur.
- Something with a probability of 1 will always occur.
- You cannot have a probability outside this range!
- Note that when somebody says it has a "100% probability"
  - That means it has a probability of 1
    - $100\% = \frac{100}{100} = 1$ ("100% probability")
- $85\% = \frac{85}{100} = 0.85$ ("85% probability")

Dice probability

What is the probability of getting a 7 by rolling two dice?

- There are six combinations that can yield 7: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)



- Thus, $|E| = 6$, $|S| = 36$,

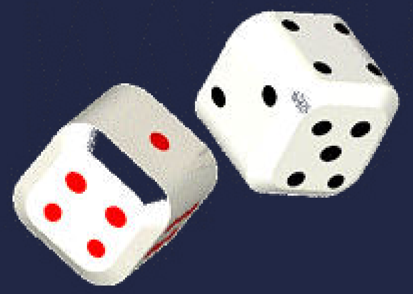
http://www.animatedimages.org/img-animated-dice-image-0079-120779.htm

$$p(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

Probability

Which is more likely:

- Rolling an 8 when 2 dice are rolled?
- Rolling an 8 when 3 dice are rolled?



- How do we approach the solution?

Probability

What is the probability of a total of 8 when 2 dice are rolled?

What is the size of the sample space? 36

How many rolls satisfy our property of interest?

2+6, 3+5, 4+4, 5+3, 6+2 5

So the probability is $\frac{5}{36} \approx 0.139$.

Probability

What is the probability of a total of 8 when 3 dice are rolled?

What is the size of the sample space? $6 \times 6 \times 6 = \mathbf{216}$

How many rolls satisfy our condition (total of 8)?

1+1+6, 1+6+1, 6+1+1 (3)

2+3+3, 3+2+3, 3+3+2 (3)

4+3+1, 4+1+3, 1+3+4, 1+4+3, 3+1+4, 3+4+1 (6)

1+2+5, 1+5+2, 2+1+5, 2+5+1, 5+1+2, 5+2+1 (6)

2+2+4, 4+2+2, 2+4+2 (3)

Total (3+3+6+6+3=21)

So the probability is **21/216 ≈ 0.097**.

Probability

Which is more likely:

- Rolling an 8 when 2 dice are rolled?
  - $5/36 \approx 0.139$
- Rolling an 8 when 3 dice are rolled?
  - $21/216 \approx 0.097$
- Rolling an 8 when 2 dice are rolled is more likely than rolling an 8 when 3 dice are rolled
  - The bigger probability is the more likely

For 3 dice?

Total of (we need to consider all different permutations)

need to consider all different permutations for each

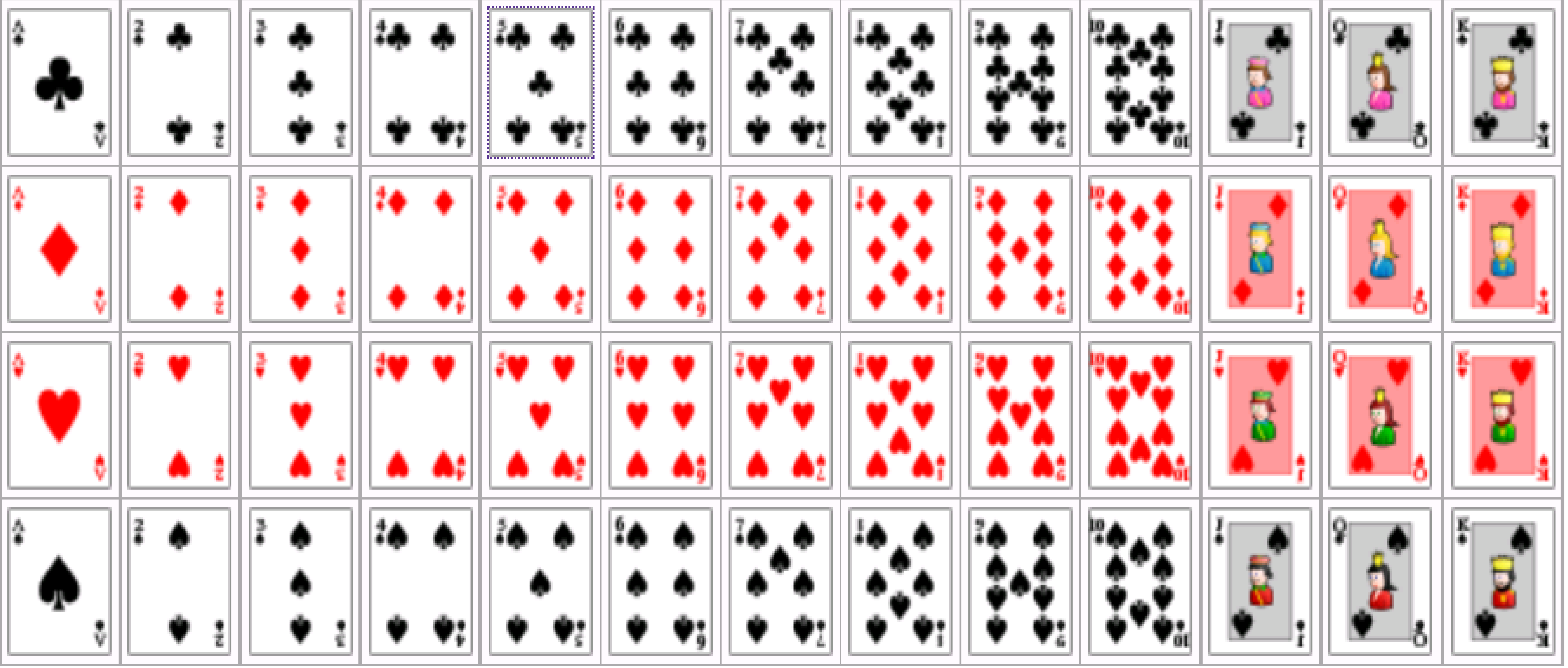
3 = 1 + 1 + 1
4 = 1 + 1 + 2
5 = 1 + 1 + 3 = 2 + 2 + 1
6 = 1 + 1 + 4 = 1 + 2 + 3 = 2 + 2 + 2
7 = 1 + 1 + 5 = 2 + 2 + 3 = 3 + 3 + 1 = 1 + 2 + 4
8 = 1 + 1 + 6 = 2 + 3 + 3 = 4 + 3 + 1 = 1 + 2 + 5 = 2 + 2 + 4
9 = 6 + 2 + 1 = 4 + 3 + 2 = 3 + 3 + 3 = 2 + 2 + 5 = 1 + 3 + 5 = 1 + 4 + 4
10 = 6 + 3 + 1 = 6 + 2 + 2 = 5 + 3 + 2 = 4 + 4 + 2 = 4 + 3 + 3 = 1 + 4 + 5
11 = 6 + 4 + 1 = 1 + 5 + 5 = 5 + 4 + 2 = 3 + 3 + 5 = 4 + 3 + 4 = 6 + 3 + 2
12 = 6 + 5 + 1 = 4 + 3 + 5 = 4 + 4 + 4 = 5 + 2 + 5 = 6 + 4 + 2 = 6 + 3 + 3
13 = 6 + 6 + 1 = 5 + 4 + 4 = 3 + 4 + 6 = 6 + 5 + 2 = 5 + 5 + 3
14 = 6 + 6 + 2 = 5 + 5 + 4 = 4 + 4 + 6 = 6 + 5 + 3
15 = 6 + 6 + 3 = 6 + 5 + 4 = 5 + 5 + 5
16 = 6 + 6 + 4 = 5 + 5 + 6
17 = 6 + 6 + 5
18 = 6 + 6 + 6

A card game

Not necessary to know!

Example set of 52 poker playing cards; 13 of each suit clubs, diamonds, hearts, and spades

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs:													
Diamonds:													
Hearts:													
Spades:													



A card game

Not necessary to know!

You are given 5 cards (this is 5-card stud poker)

The goal is to obtain the best hand you can

The possible poker hands are (in increasing order):

- No pair
- One pair (two cards of the same face)
- Two pair (two sets of two cards of the same face)
- Three of a kind (three cards of the same face)
- Straight (all five cards sequentially – ace is either high or low)
- Flush (all five cards of the same suit)
- Full house (a three of a kind of one face and a pair of another face)
- Four of a kind (four cards of the same face)
- Straight flush (both a straight and a flush)
- Royal flush (a straight flush that is 10, J, K, Q, A)

Card game probability: royal flush

What is the chance of getting a royal flush?

- That's the cards 10, J, Q, K, and A of the same suit

There are only 4 possible royal flushes.

Possibilities for 5 cards: C(52, 5) = 2, 598, 960

Probability = 4/2, 598, 960 = 0.0000015

- **Or about 1 in 650, 000**

Card game probability: four of a kind

What is the chance of getting 4 of a kind when dealt 5 cards?

- Possibilities for 5 cards: C(52, 5) = 2, 598, 960

Possible hands that have four of a kind:

- There are 13 possible four of a kind hands
- The fifth card can be any of the remaining 48 cards
- Thus, total possibilities is 13*48 = 624

Probability = 624/2, 598, 960 = 0.00024

- Or 1 in 4165

Card game probability: flush

What is the chance of getting a flush?

- That's all 5 cards of the same suit

We must do ALL of the following:

- Pick the suit for the flush: C(4, 1)
- Pick the 5 cards in that suit: C(13, 5)

As we must do all of these, we multiply

the values out (via the product rule)

This yields $\binom{13}{5}\binom{4}{1}$=5148

Possibilities for 5 cards: C(52, 5) = 2, 598, 960

Probability = 5148/2, 598, 960 = 0.00198

- Or about 1 in 505

Note that if you don't count straight flushes (and thus royal flushes) as a "flush", then the number is really 5108

Poker probability: full house

What is the chance of getting a full house?

- That's three cards of one face and two of another face

We must do ALL of the following:

- Pick the face for the three of a kind: C(13, 1)
- Pick the 3 of the 4 cards to be used: C(4, 3)
- Pick the face for the pair: C(12, 1)
- Pick the 2 of the 4 cards of the pair: C(4, 2)

As we must do all of these, we multiply the values out (via the product rule)

This yields $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$

Possibilities for 5 cards: C(52, 5) = 2, 598, 960

Probability = 3744/2, 598, 960 = 0.00144

- Or about 1 in 694

Inclusion-exclusion principle

The possible card game hands are (in increasing order):

Not necessary to know!

• Nothing	
• One pair	cannot include two pair, three of a kind, four of a kind, or full house
• Two pair	cannot include three of a kind, four of a kind, or full house
• Three of a kind	cannot include four of a kind or full house
• Straight	cannot include straight flush or royal flush
• Flush	cannot include straight flush or royal flush
• Full house	
• Four of a kind	
• Straight flush	cannot include royal flush
• Royal flush	

Card game: three of a kind

What is the chance of getting a three of a kind?

- That's three cards of one face
- Can't include a full house or four of a kind

We must do ALL of the following:

- Pick the face for the three of a kind: C(13, 1)
- Pick the 3 of the 4 cards to be used: C(4, 3)
- Pick the two other cards' face values: C(12, 2)
    - We can't pick two cards of the same face!
- Pick the suits for the two other cards: C(4, 1)*C(4, 1)

As we must do all of these, we multiply the values out (via the product rule)

This yields $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54912$

Possibilities for 5 cards: C(52, 5) = 2, 598, 960

Probability = 54, 912/2, 598, 960 = 0.0211 Or about 1 in 47

Card game hand odds

Not necessary to know!

The possible poker hands are (in increasing order):

• Nothing	1, 302, 540	0.5012
• One pair	1, 098, 240	0.4226
• Two pair	123, 552	0.0475
• Three of a kind	54, 912	0.0211
• Straight	10, 200	0.00392
• Flush	5, 108	0.00197
• Full house	3, 744	0.00144
• Four of a kind	624	0.000240
• Straight flush	36	0.0000139
• Royal flush	4	0.00000154

Event Probabilities

Let E be an event in a sample space S. The probability of the complement of E is:

$$p(\bar{E}) = 1 - p(E)$$

Recall the probability for getting a royal flush is 0.0000015

- The probability of *not* getting a royal flush is 1 - 0.0000015 = 0.9999985

Recall the probability for getting a four of a kind is 0.00024

- The probability of *not* getting a four of a kind is 1 - 0.00024 = 0.99976

Probability of the union of two events

Let E_1 and E_2 be events in sample space S

Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

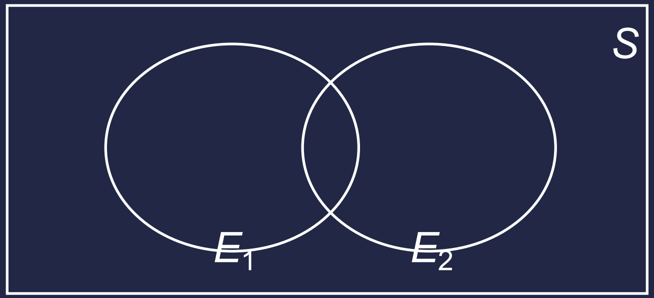
$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$p\,(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|}$$

$$= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2)$$



Probability of the union of two events

If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?

Let *n* be the number chosen

- p (2 div n) = 50/100 (all the even numbers)
- p (5 div n) = 20/100
- p (2 div n) and p (5 div n) =
  p (10 div n) = 10/100
- p (2 div n) or p (5 div n) =
  p (2 div n) + p (5 div n) - p (10 div n)
  = 50/100 + 20/100 – 10/100
  = 3/5

Monty Hall Puzzle

Monty Hall Puzzle

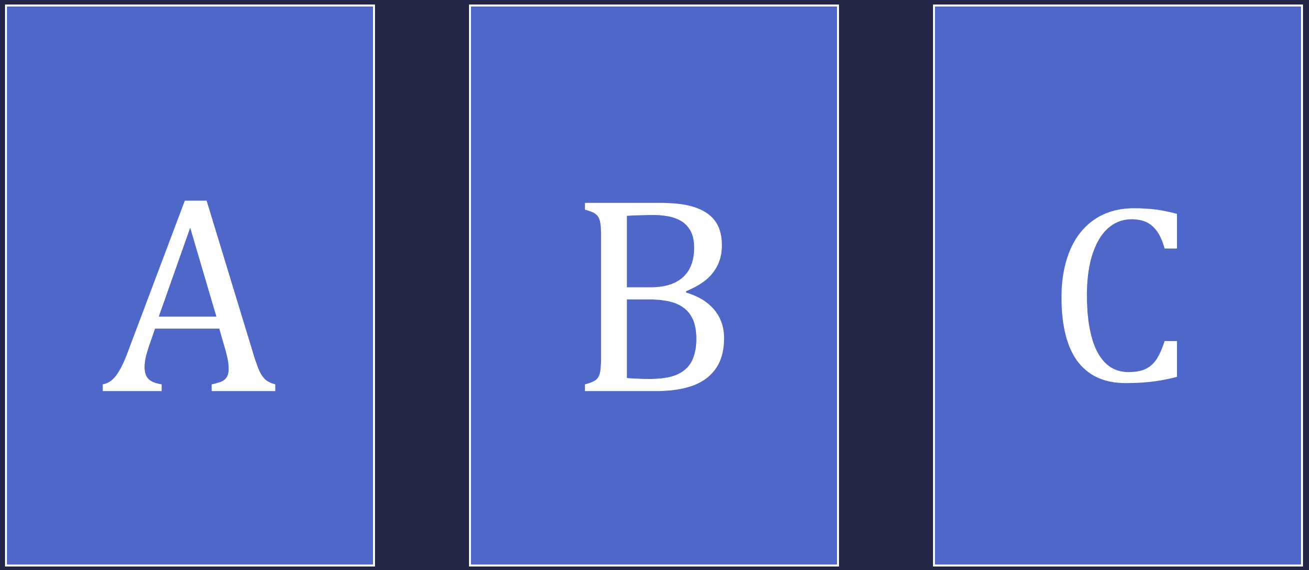
Choose a door to
win a prize!

Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what is behind the doors, opens another door, say No. 2, which has a goat. He then says to you, "Do you want to pick door No. 3?“

Is it to your advantage to switch your choice? If so, why? If not, why not?

The Monty Hall Problem

Pick a door:



The Monty Hall Problem

Pick a door:

A

B

C

The Monty Hall Problem

B

C

After opening this door, would you like to change?

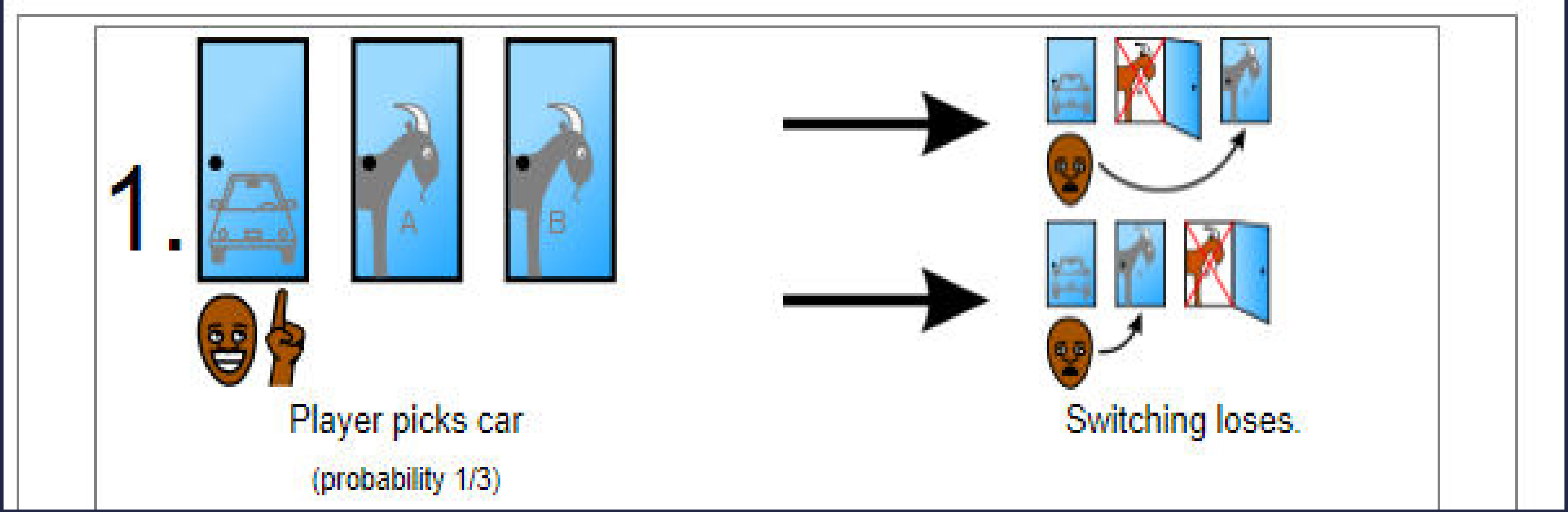
The Monty Hall Problem

B

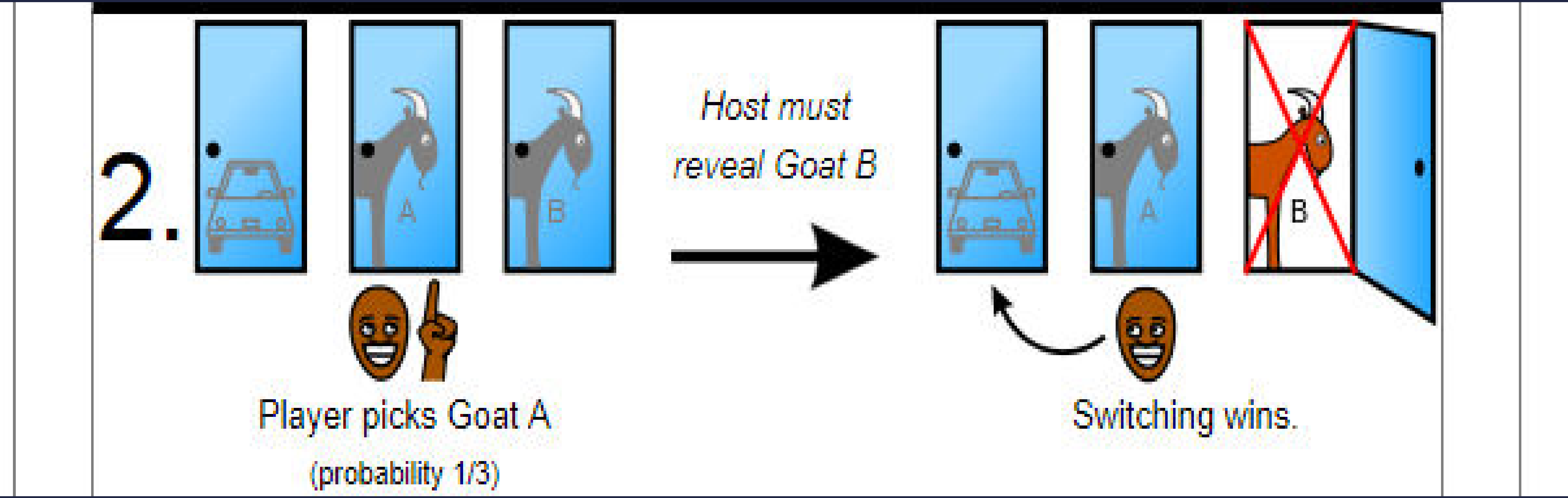
C?

Would you like to switch?

Monty Hall Puzzle http://en.wikipedia.org/wiki/Monty_Hall_problem



Monty Hall Puzzle http://en.wikipedia.org/wiki/Monty_Hall_problem



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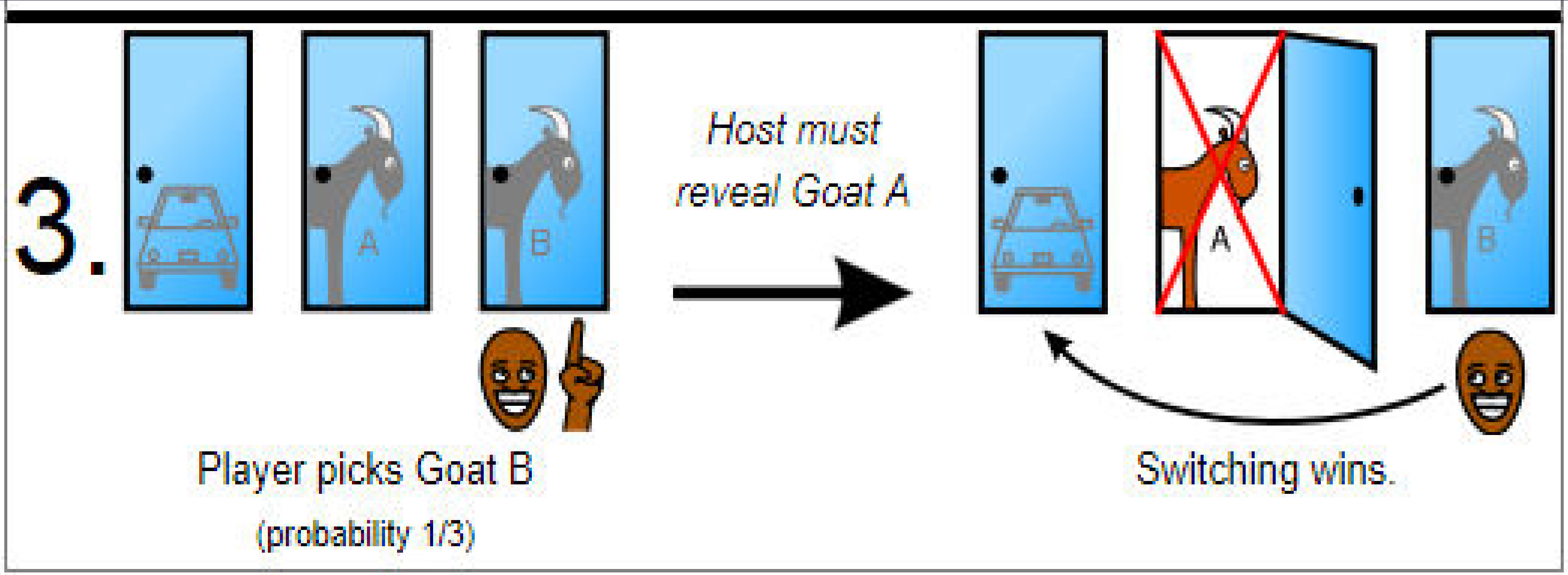
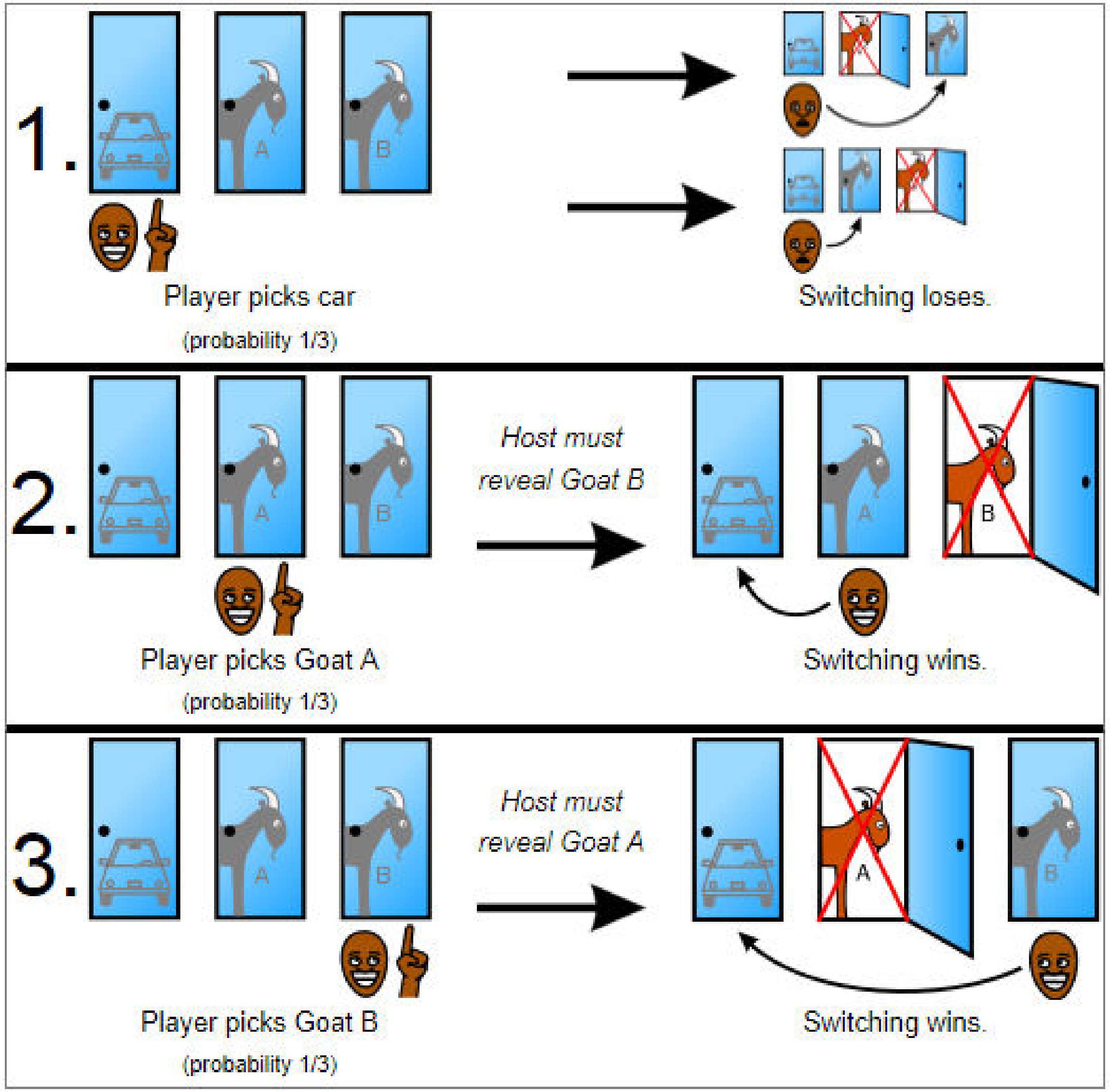


The player has an equal chance of initially selecting the car, Goat A, or Goat B. Switching results in a win 2/3 of the time.

Monty Hall Puzzle http://en.wikipedia.org/wiki/Monty_Hall_problem

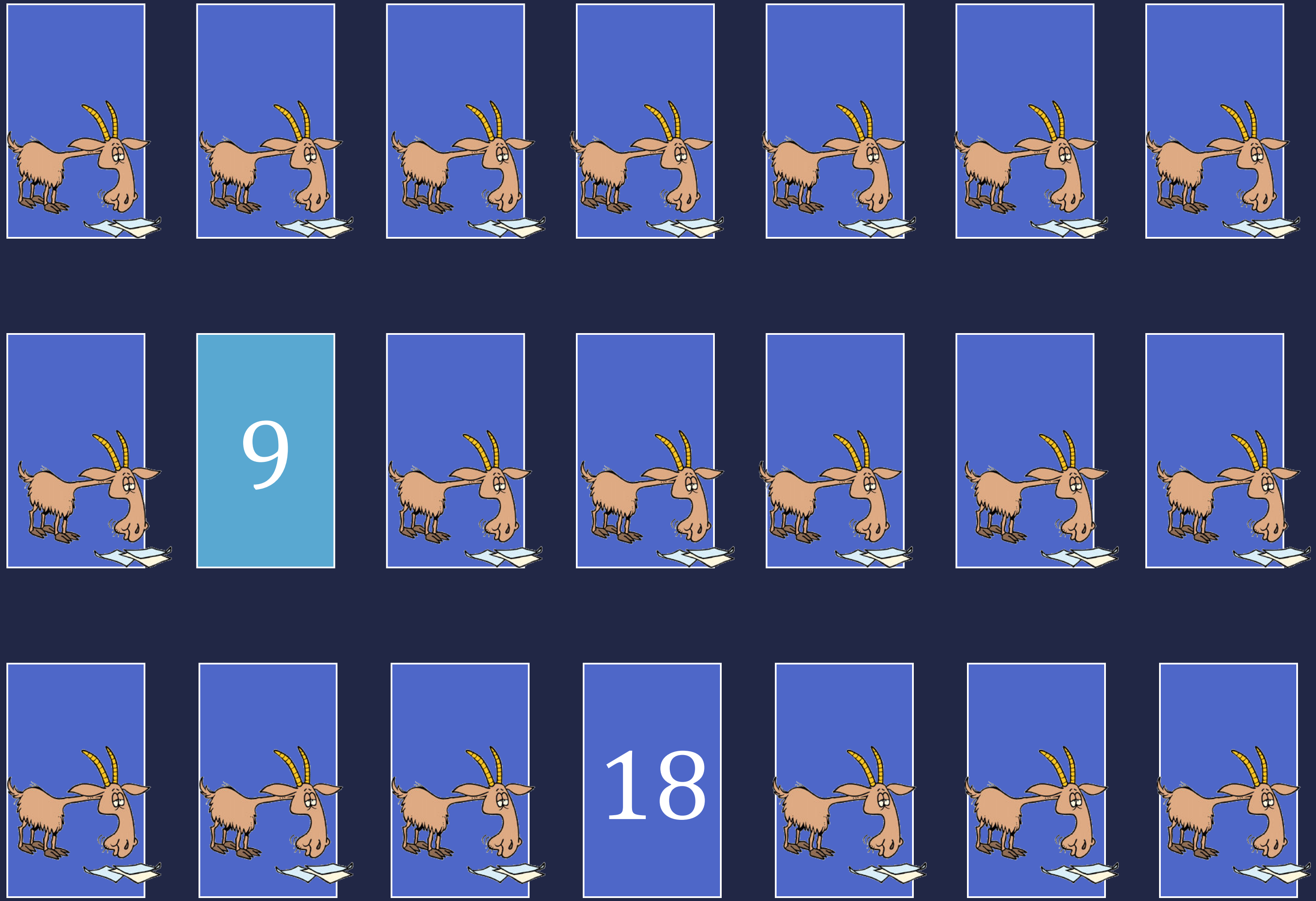


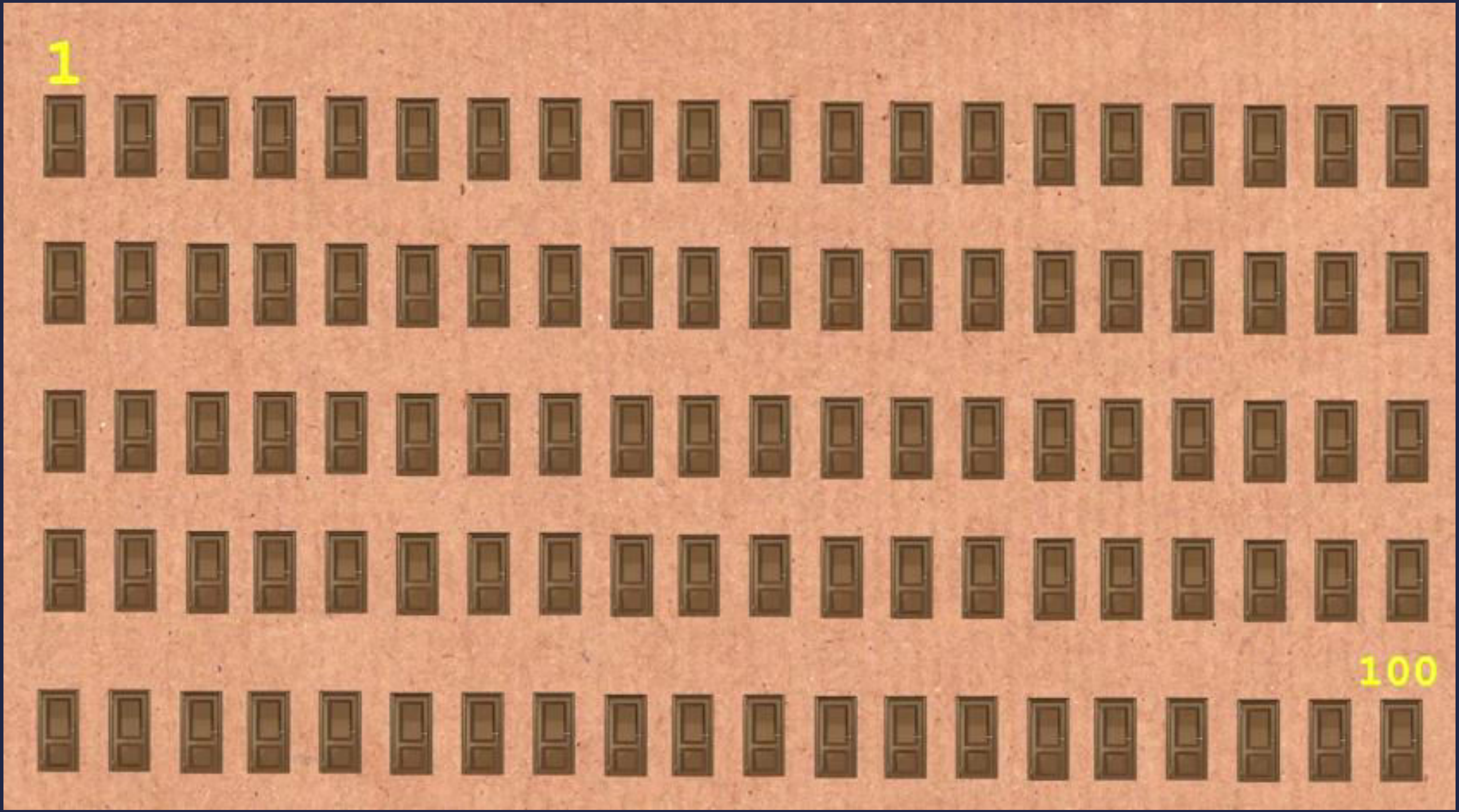
The player has an equal chance of initially selecting the car, Goat A, or Goat B. Switching results in a win 2/3 of the time.

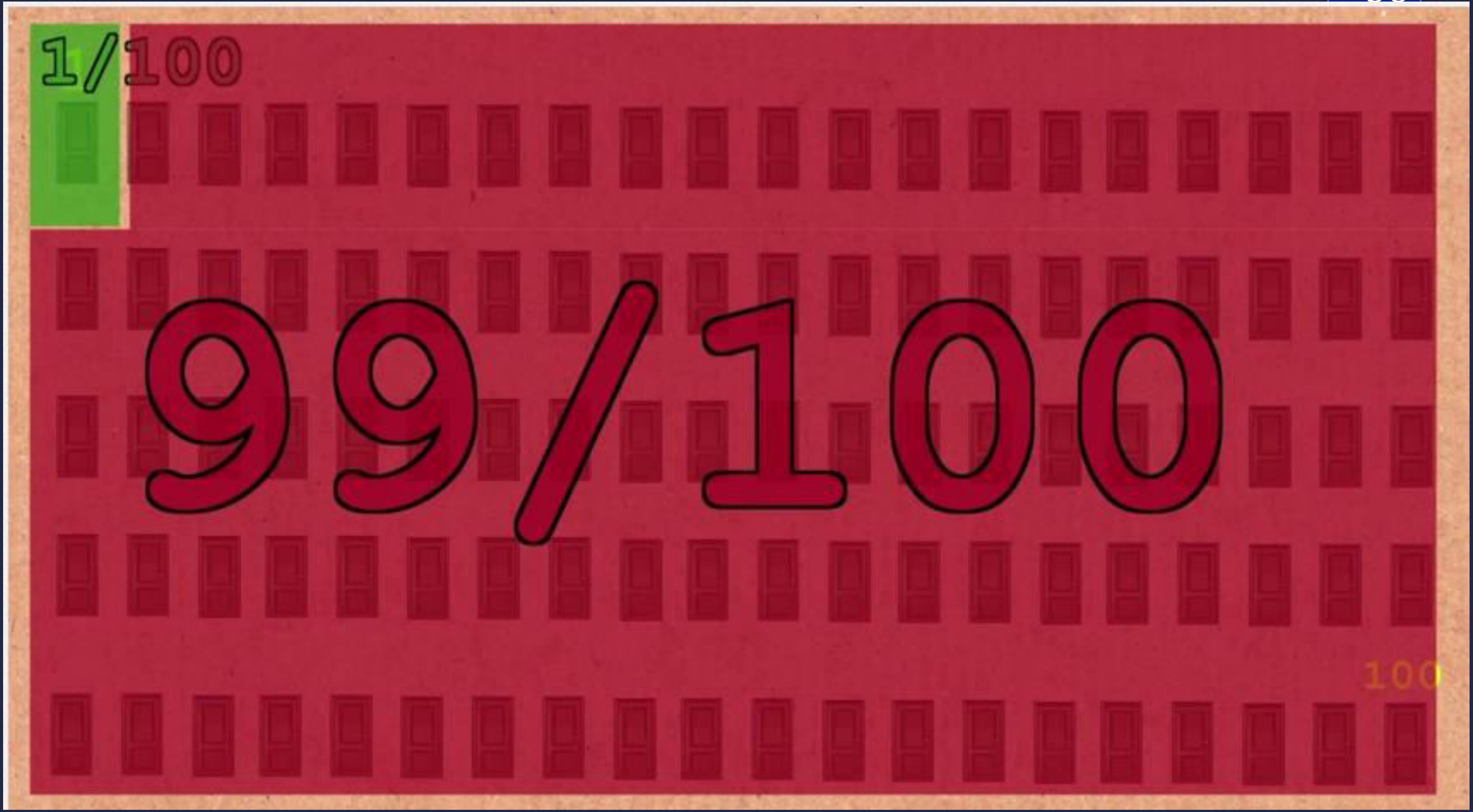
Switching results in a win 2/3 of the time

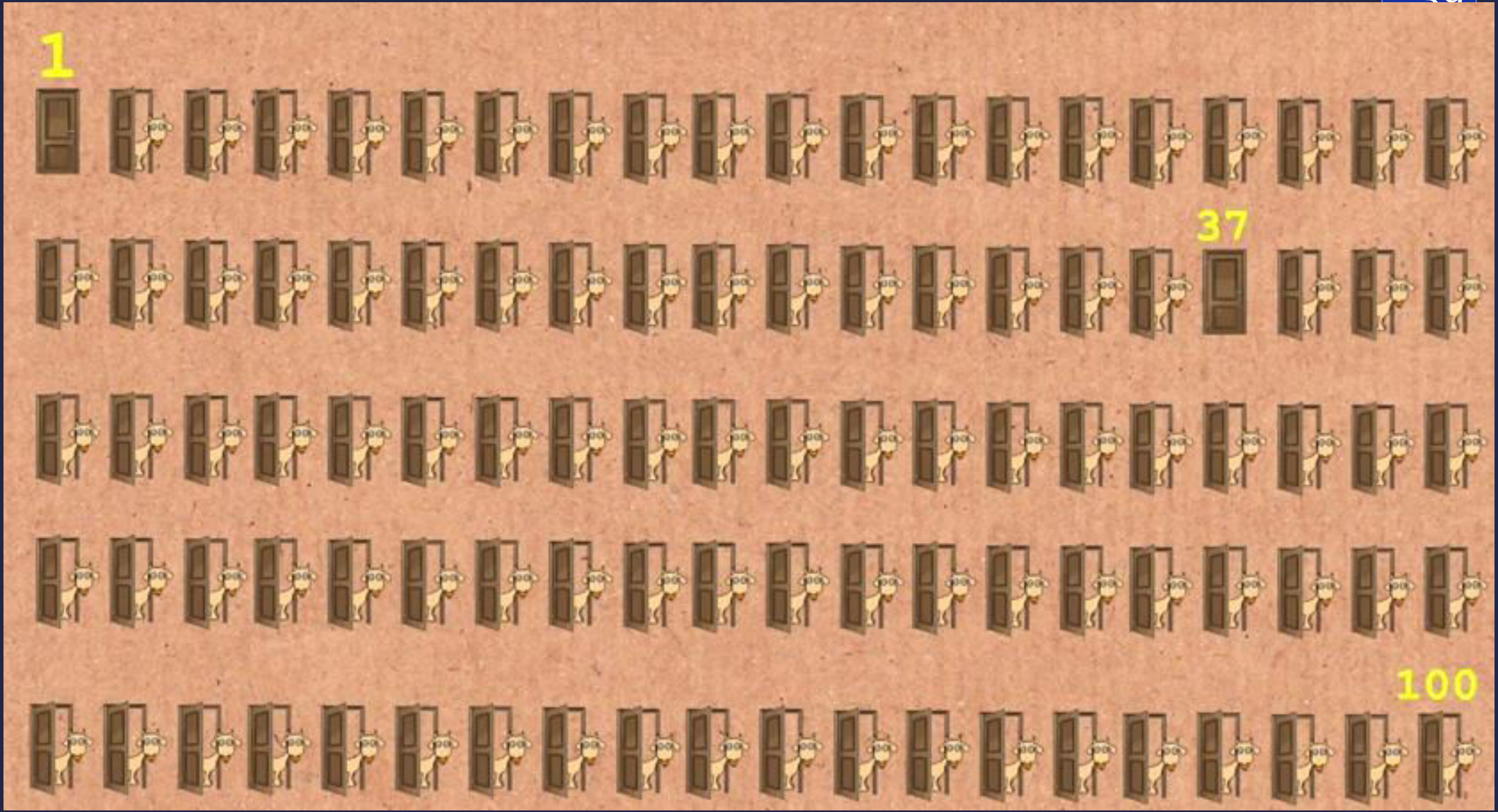
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21


9
18


1
100


1/100
99/100
100


1
37
100

Check this
https://www.youtube.com/playlist?list=PLt5AfwLFPxWLzNG4Ttv9-qZj86xt-J9_W