2.3 Functions

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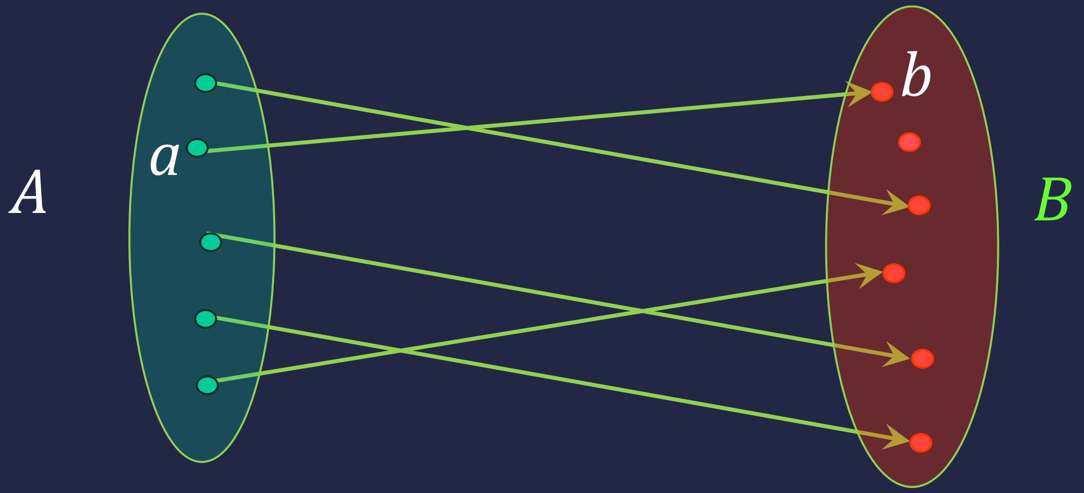
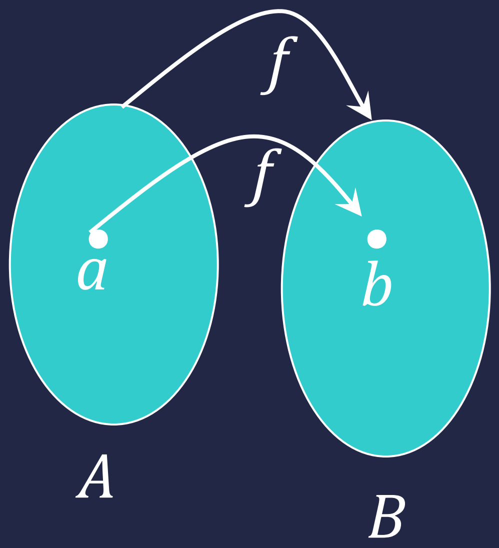
Definition of Functions

- Given any sets A, B, a function f from *(or "mapping") A to B* ($f : A \rightarrow B$) is an assignment of **exactly one** element $\boldsymbol{f(x) \in B}$ to each element $\boldsymbol{x \in A}$

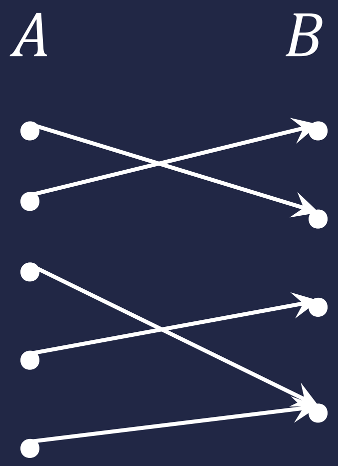


Graphical Representations

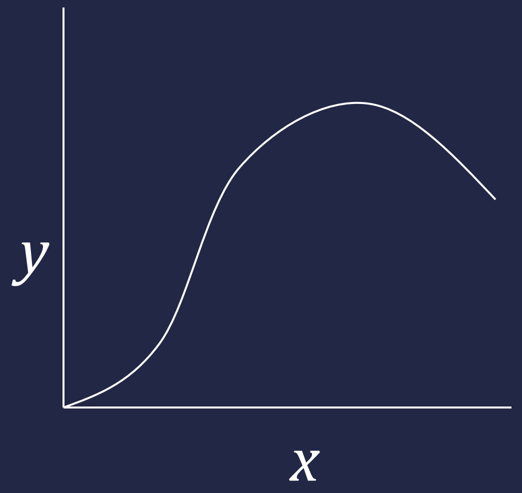
- Functions can be represented graphically in several ways:



Like Venn diagrams



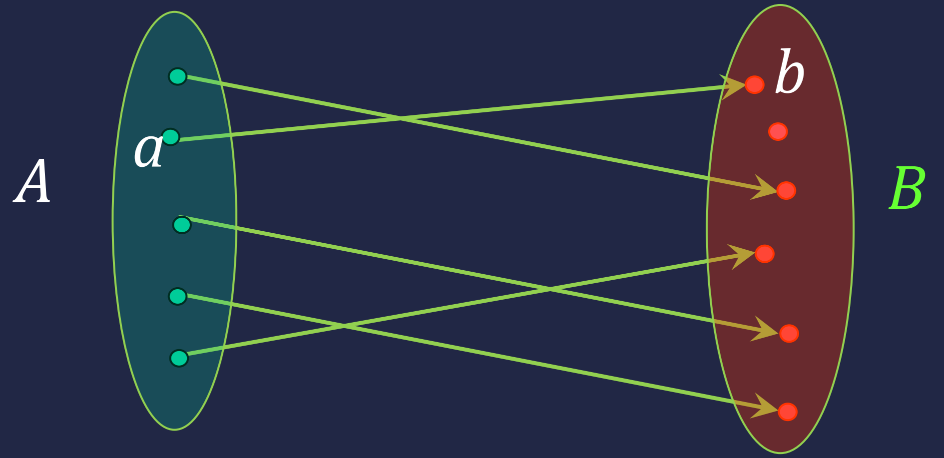
Graph



Plot

Terminology: domain – codomain, image – preimage

- If $f : A \to B$, and $f(a) = b$ (where $a \in A$ & $b \in B$), then:
  - A is the *domain* of f
  - B is the *codomain* of f
  - b is the *image* of a under f
  - a is a *pre-image* of b under f
    - In general, b may have more than one pre-image
  - The *range* $R \subseteq B$ of f is $\{b \mid \exists a\, f(a) = b\}$



Range vs. Codomain - Example

- Suppose that: "*f is a function mapping students in this class to the set of grades* {A+, A, B+, B, C+, C, D+, D, F, DN}"
- At this point, you know *f*'s codomain is:

  {A+, A, B+, B, C+, C, D+, D, F, DN}

and its range is unknown!

- Suppose the grades turn out all A+'s and B+'s.

Then the range of *f* is {A+, B+},

but its codomain is still {A+, A, B+, B, C+, C, D+, D, F, DN}

Function Addition/Multiplication

- We can add and multiply *functions*

$f, g: \mathbf{R} \to \mathbf{R}$:

$(f+g): \mathbf{R} \to \mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$

$(f \times g): \mathbf{R} \to \mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Images of Sets under Functions

- Given $f: A \rightarrow B$, and $S \subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{b \mid \exists s \in S: f(s) = b\}$
  $:\equiv \{f(s) \mid s \in S\}$. (shorthand)
- Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$, and $f(e) = 1$.
  What is the image of the subset $S = \{b, c, d\}$
- Solution: It is the set $f(S) = \{1, 4\}$

One-to-One Functions

- A function is *one-to-one* (1-1), or *injective*, or *an injection*, iff every element of its range has **only one** pre-image.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?

May Be Larger

One-to-One Functions

- Given $f : A \rightarrow B$

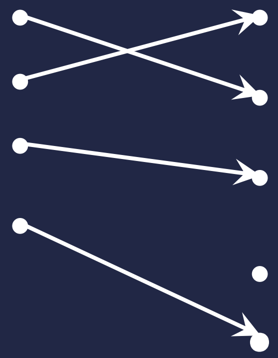
"f is injective" $\equiv (\forall x, y: (x \neq y) \rightarrow (f(x) \neq f(y)))$

or

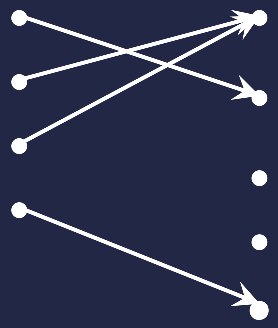
"f is injective" $\equiv (\forall x, y: (f(x) = f(y)) \rightarrow (x = y))$

One-to-One Illustration

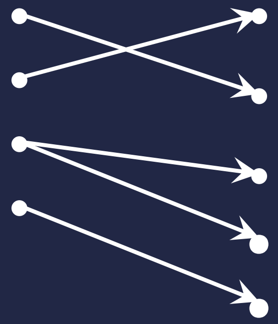
- Graph representations of functions that are (or not) one-to-one:



One-to-one



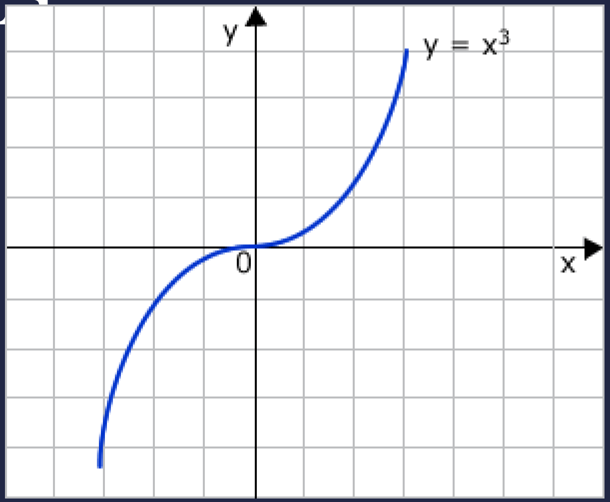
Not one-to-one



Not even a function!

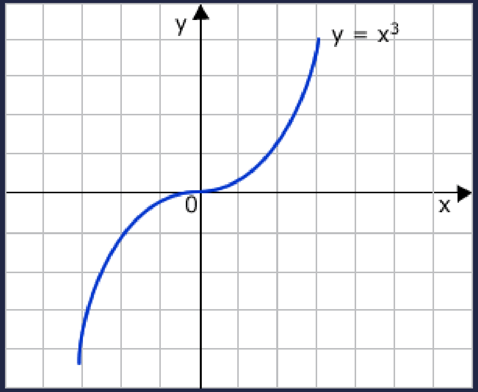
strictly increasing and strictly decreasing functions are 1-1

- Definitions (for functions f over numbers):
  - f is *strictly increasing* iff $x > y \to f(x) > f(y)$ for all x, y in domain;
  - f is *strictly decreasing* iff $x > y \to f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strict decreasing, then f is one-to-one.
  - *e.g.* $f(x) = x^3$



Onto (Surjective) Functions

- A function $f: A \rightarrow B$ is *onto* or *surjective* or *a surjection* **iff its range is equal to its codomain** $(\forall b \in B, \exists a \in A: f(a) = b)$.
- An *onto* function maps the set A onto (over, covering) the *whole* set B, not just over a piece of it.
  - *e.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)



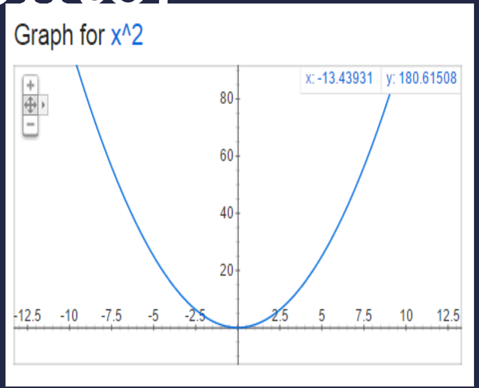
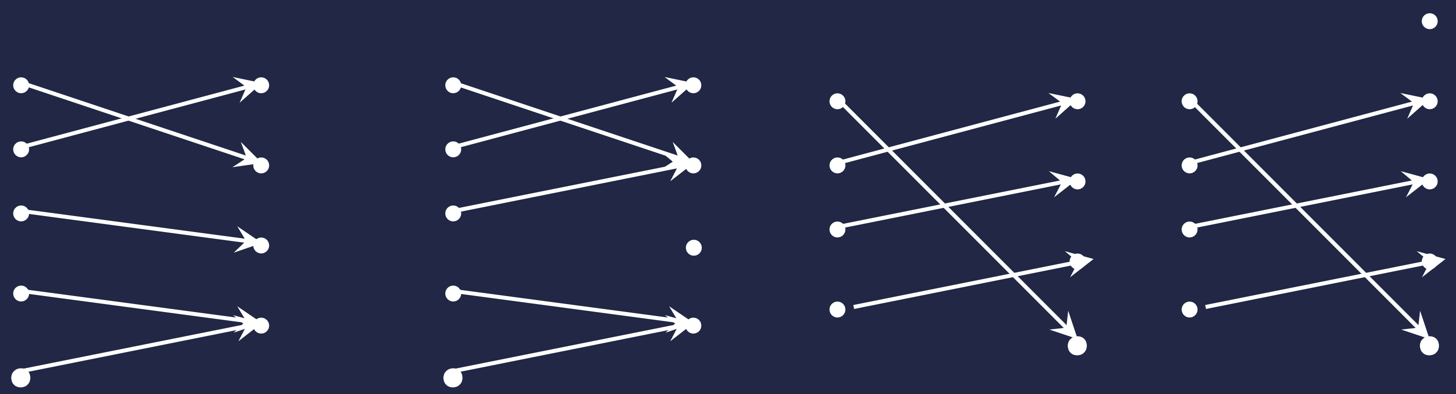


Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Onto
(Not 1-1)

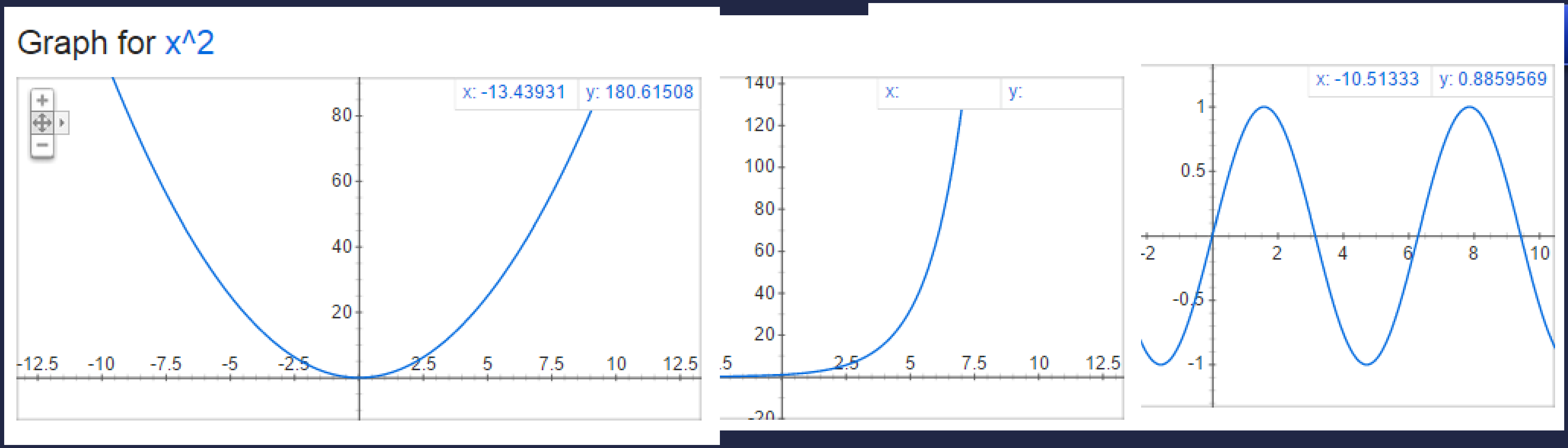
Not Onto
(Not 1-1)

Both 1-1
and onto

1-1 but
not onto

Bijections

- A function f is
  a one-to-one correspondence, or
  a bijection, or
  reversible, or *invertible*,
  iff **it is both one-to-one and onto**.



Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x) = \sin(x)$	Real	Real	No	No	No
$f(x) = 2^x$	Real	Positive real	Yes	Yes	Yes
$f(x) = x^2$	Real	Positive real	No	Yes	No
Reverse string	Bit strings of length n	Bit strings of length n	Yes	Yes	Yes

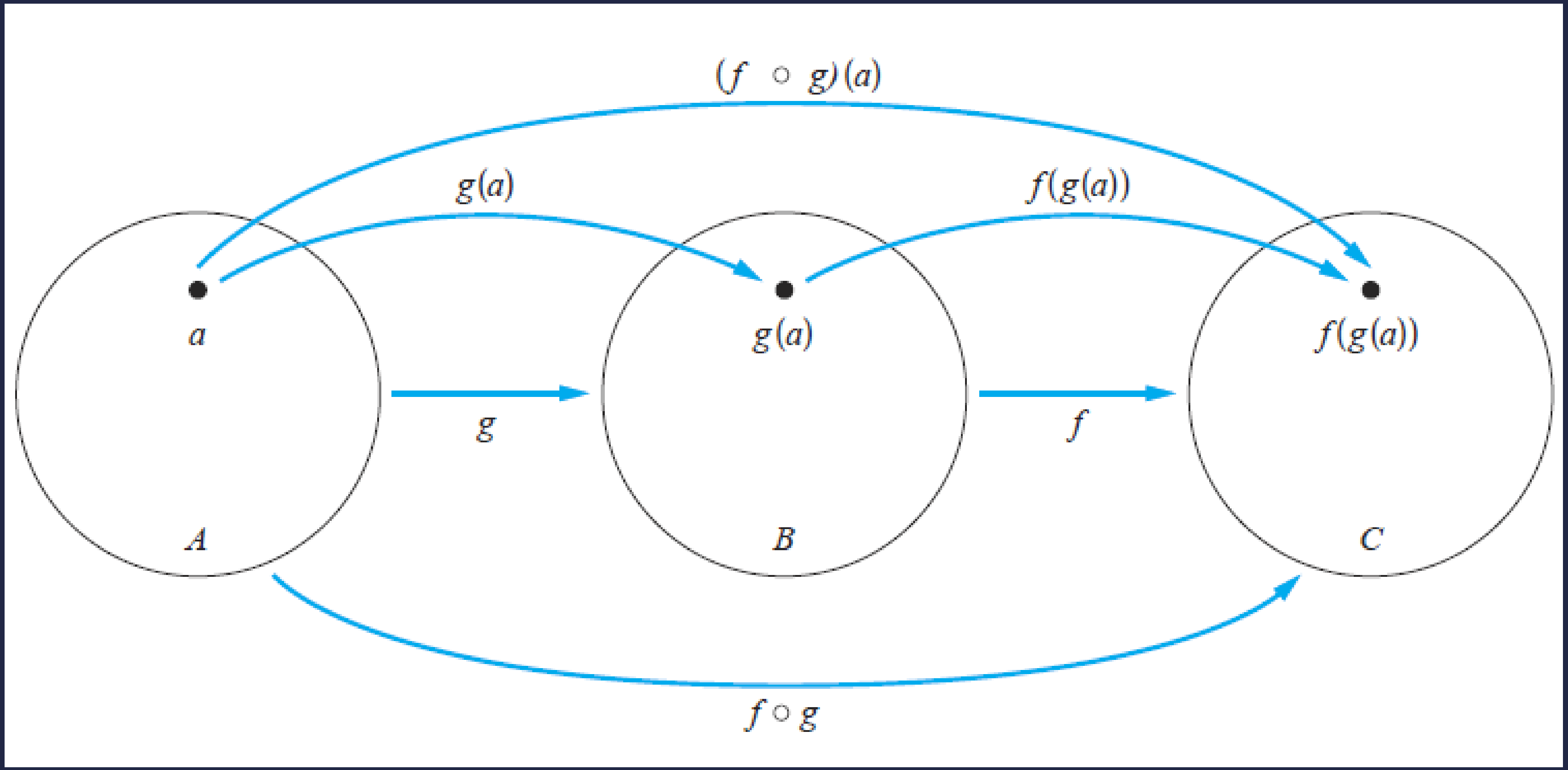
Function Composition

- For functions $g: A \rightarrow B$ and $f: B \rightarrow C$, there is a special operator called *compose* ("$\circ$").
  - It composes (i.e., creates) a new function out of f, g by applying f to the result of g

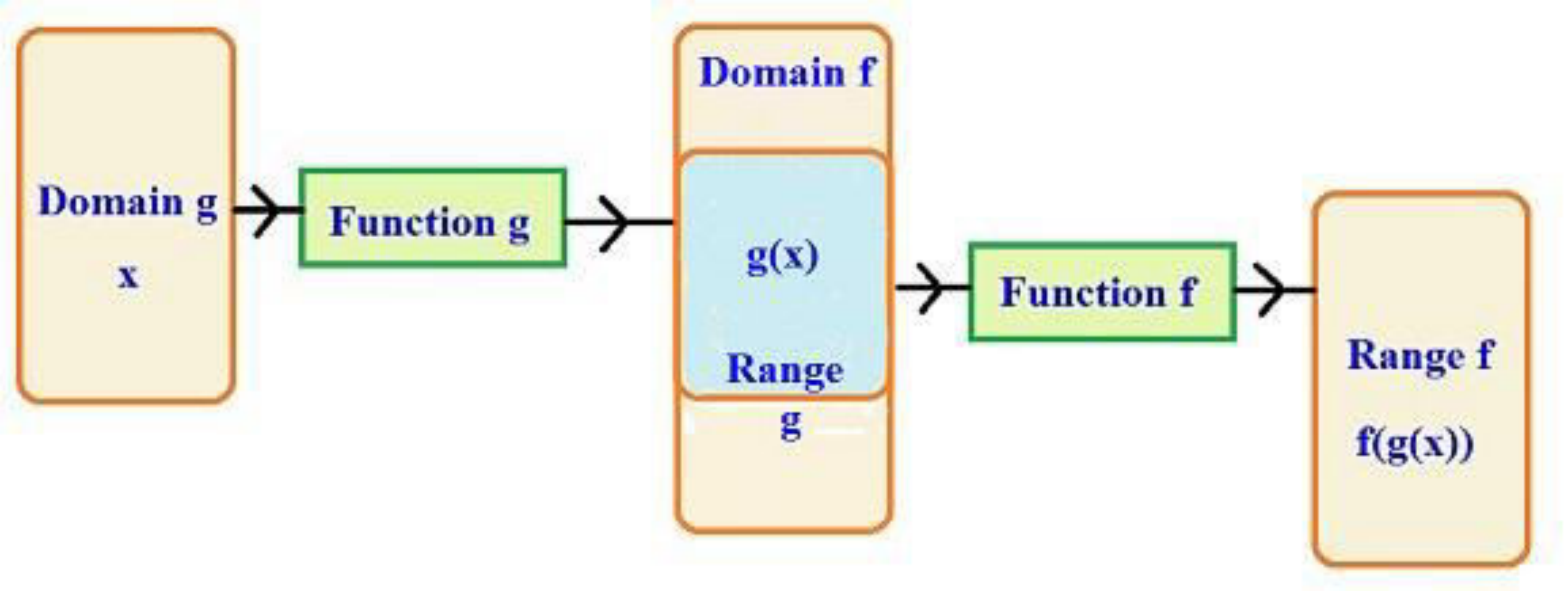
$$(f \circ g): A \rightarrow C, \text{ where } (f \circ g)(a) = f(g(a))$$

  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$
  - **The range of g must be a subset of f's domain!!**
  - Note that $\circ$ (like Cartesian $\times$, but unlike $+$, $\wedge$, $\cup$) is non-commuting. (In general, $f \circ g \neq g \circ f$.)

Function Composition

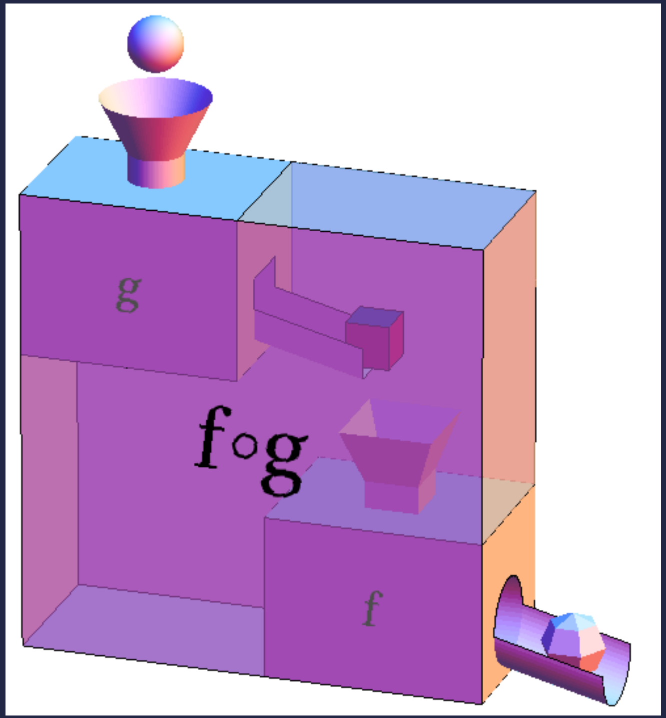


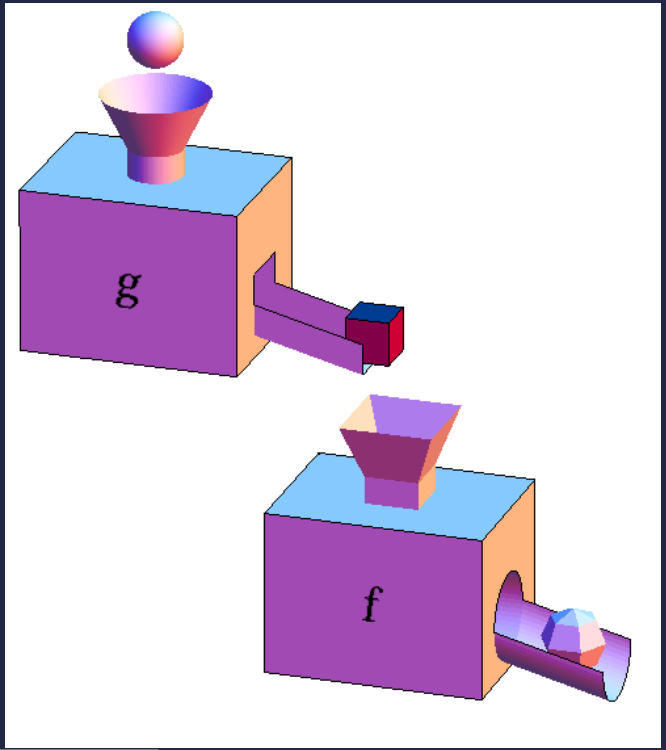
Function Composition



http://www.mathcaptain.com/algebra/composition-of-functions.html

Function Composition





http://mathinsight.org/image/function_machines_composed_combined

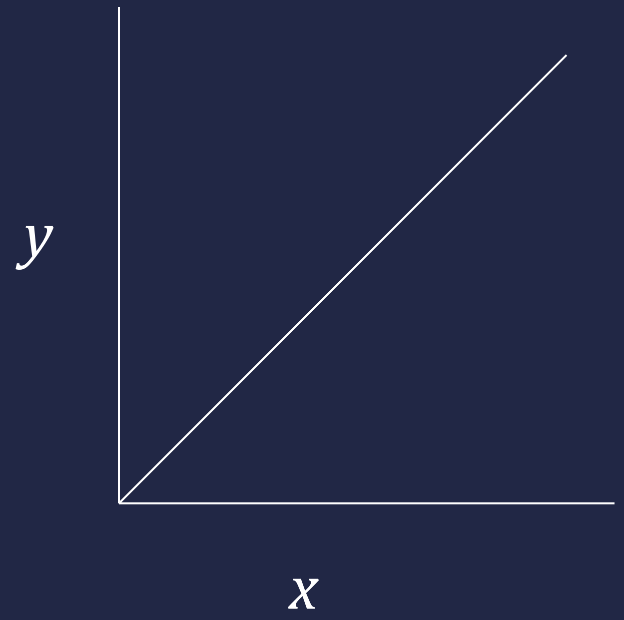
The Identity Function (I)

- For any domain A, the *identity function* $I: A \rightarrow A$ (sometimes written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a) = a$.
- Some identity functions we have seen:
  - $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with U.
    - $p \wedge \mathbf{T} = p$ $\qquad$ $p \vee \mathbf{F} = p$
    - $A \cup \varnothing = A$ $\qquad$ $A \cap U = A$
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:

Domain and range

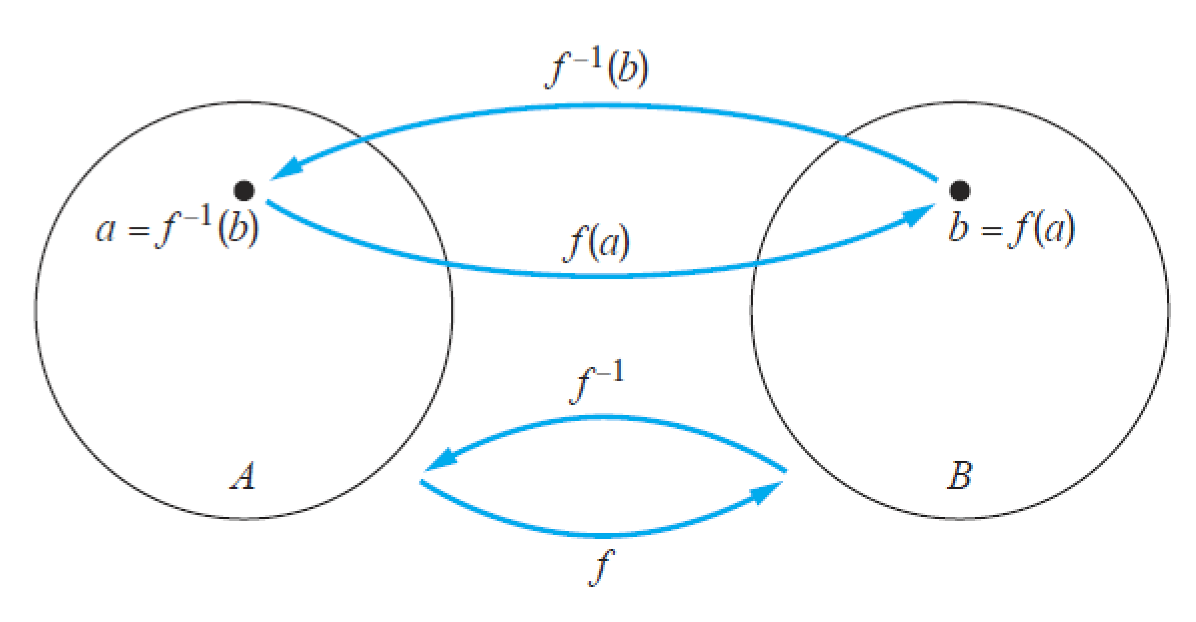


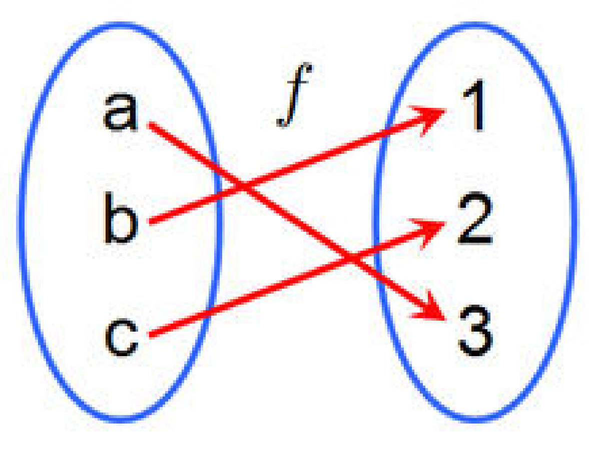
Inverse of a Function

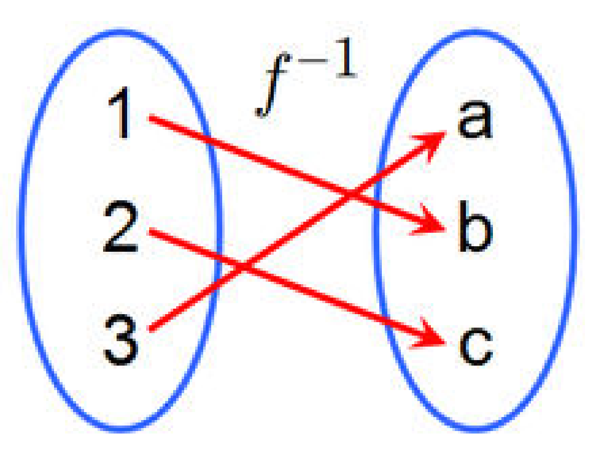
- For bijections f: $A \rightarrow B$, there exists an *inverse* of f, written f^{-1}: $B \rightarrow A$, which is the unique function such that:

$$f^{-1} \circ f = I \text{ (The identity function)}$$

Inverse of a function







Graphs of Functions

- We can represent a function $f: A \to B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only one pair $(a, f(a))$.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
  - A function is then drawn as a curve (set of points) with only one y for each x.

Graphs of Functions

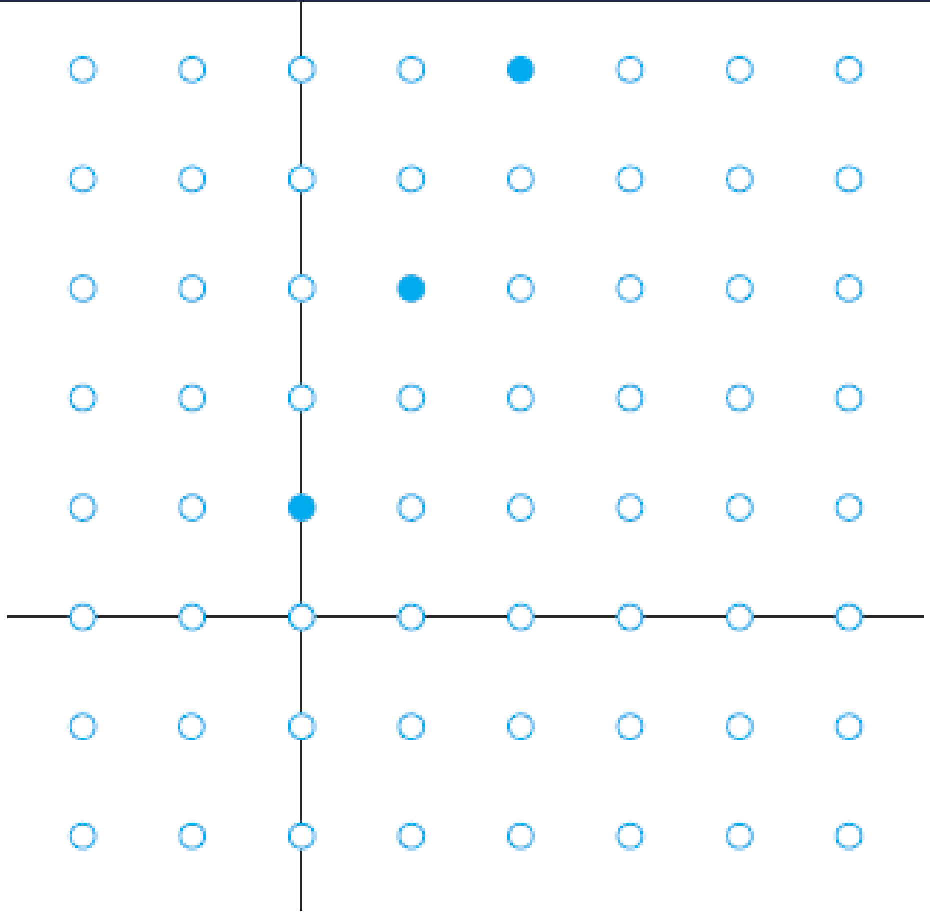
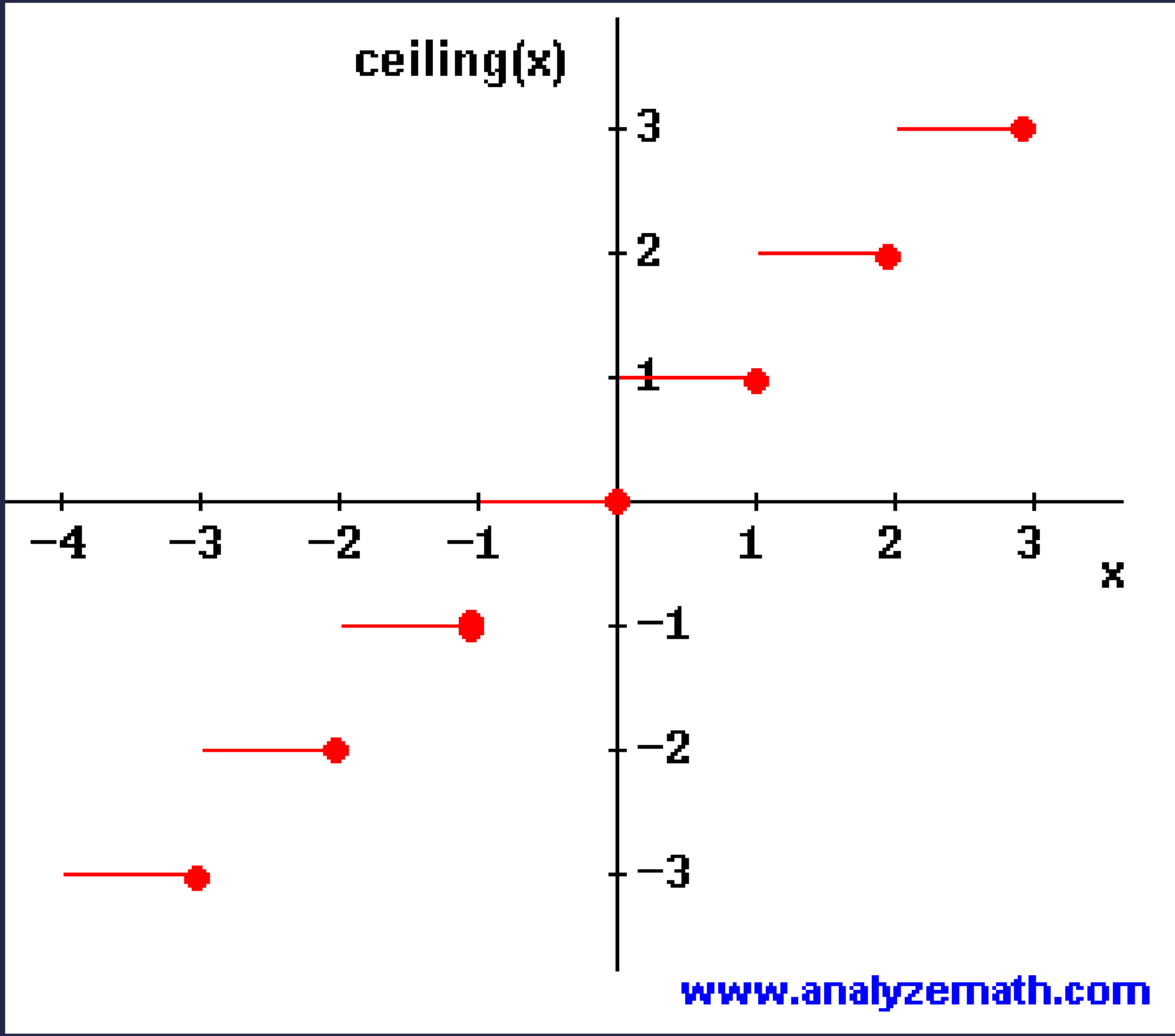


FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

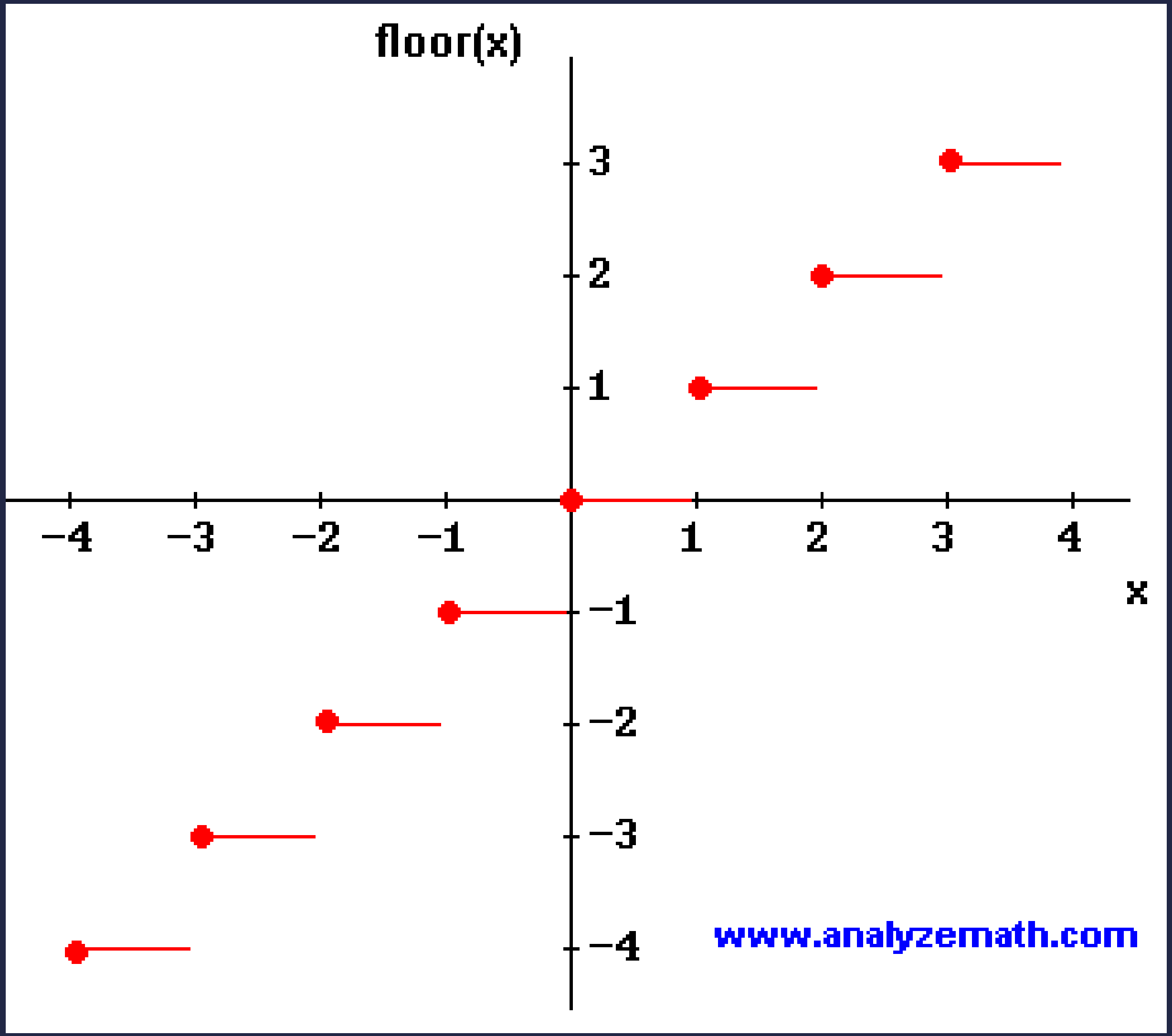
Floor and ceiling Functions

- In discrete math, we frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ (“floor of x”) is the largest integer $\leq x$.
  - $\lceil x \rceil$ (“ceiling of x”) is the smallest integer $\geq x$.

Ceiling Function $\lceil x \rceil$ ("ceiling of x")



Floor Function $\lfloor x \rfloor$ ("floor of x")



Ceiling and Floor Properties

Let x be real and n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$

(1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$

(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$

Ceiling and Floor Properties

Let x be real and n be an integer

(2) $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Visualizing Floor & Ceiling

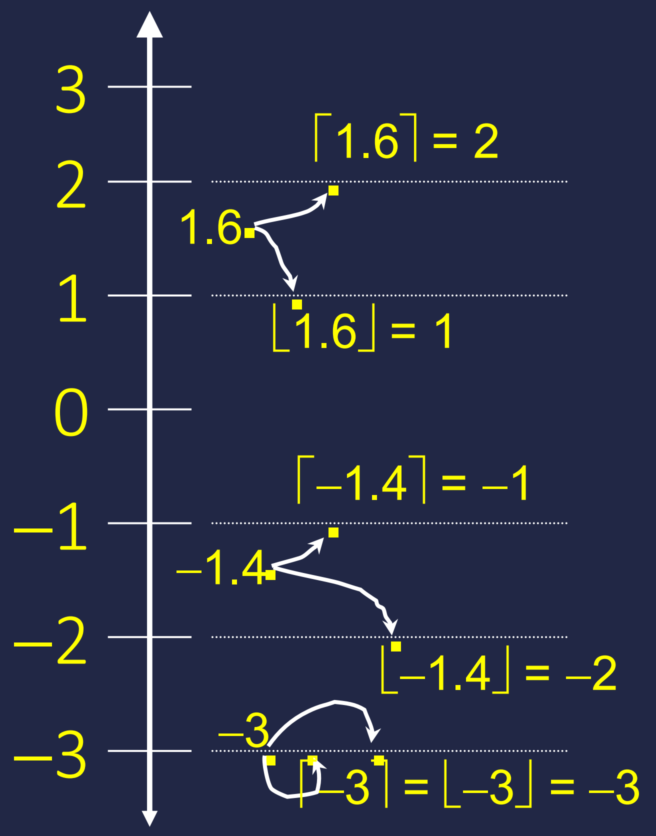
- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$,

$$\lfloor -x \rfloor \neq -\lfloor x \rfloor \text{ and}$$

$$\lceil -x \rceil \neq -\lceil x \rceil$$

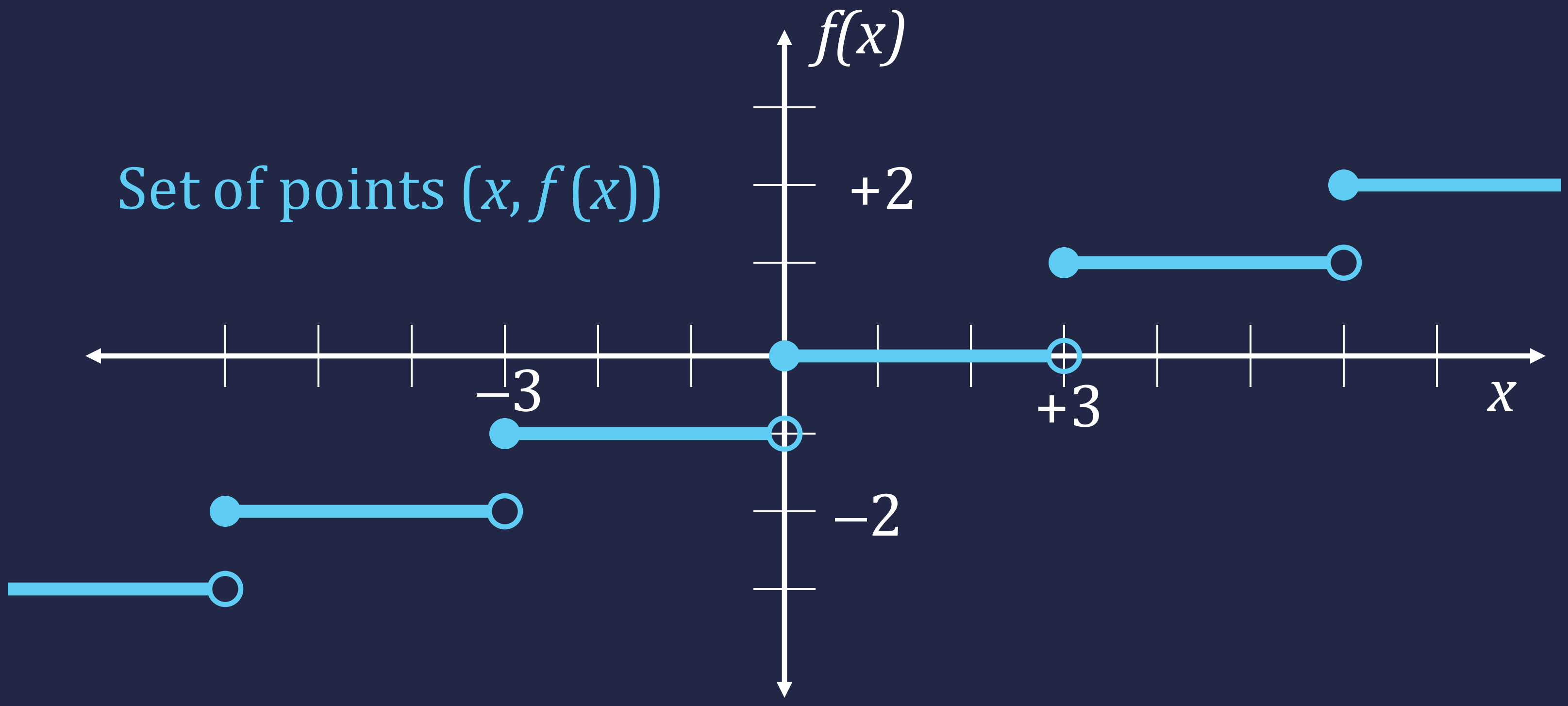
- Note that if $x \in \mathbf{Z}$,

$$\lfloor x \rfloor = \lceil x \rceil = x.$$



Example of Plots with floor/ceiling

Plot of graph of function $f(x) = \lfloor x / 3 \rfloor$



Factorial Function

- The factorial function gives the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects
- **Definition**: The factorial function, denoted $n!$, is a function $\mathbb{N} \rightarrow \mathbb{N}^{+}$. Its value is the product of the n positive integers

$$n! = \prod_{i=1}^{i=n} i = 1 \,.\, 2 \,.\, 3 \,.\, \ldots \,.\, (n-1) \,.\, n$$

Factorial Function

- $f(1) = 1! = 1$
- $f(2) = 2! = 1 . 2 = 2$
- $\vdots$
- $f(6) = 6! = 1 . 2 . 3 . 4 . 5 . 6 = 720$
- $\vdots$
- $f(20) = 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10 . 11 . 12 . 13 . 14 . 15 . 16 . 17 . 18 . 19 . 20$

  $= 2, 432, 902, 008, 176, 640, 000$