Advanced Counting Techniques

8.2 Solving Linear Recurrence Relations

Credit

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Solving Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas (formulae) in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Linear Homogeneous Recurrence Relations with constant coefficients

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$

Linear Homogeneous Recurrence Relations with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}, \; c_k \neq 0$$

Linear: The RHS is the sum of previous terms of the sequence each multiplied by a function of n. All terms a_j occur to the first power.

Homogeneous: No terms occur that are not multiplies of the a_j's.

Degree k: a_n is expressed in terms of the previous k terms of the sequence

Constant coefficients: $c_1, c_2, \dots, c_k$ are constants. They are not function of n.

8.2 Solving Linear homogeneous Recurrence Relations

- Definition: A linear homogeneous recurrence relation of degree k with constant coefficients has the form:
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$,
  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

  - Examples (A *linear homogeneous recurrence relation of degree k with constant coefficients*)
    - $P_n = (1.11)P_{n-1}$ (Degree 1) $\{c_1 = 1.11\}$
    - $f_n = f_{n-1} + f_{n-2}$ (Degree 2) $\{c_1 = 1, c_2 = 1\}$
    - $a_n = a_{n-5}$ (Degree 5) $\{c_1 = c_2 = c_3 = c_4 = 0, c_5 = 1\}$
    - $a_n = a_{n-1} + a_{n-3}$ (Degree 3) $\{c_1 = 1, c_2 = 0, c_3 = 1\}$

8.2 Solving Linear homogeneous Recurrence Relations

- Definition: A linear homogeneous recurrence relation of degree k with constant coefficients has the form:
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$,
  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

  - Examples (Not A linear homogeneous recurrence relation of degree k with constant coefficients)

    $a_n = a_{n-1} + (a_{n-2})^2$ (not liner)

    $H_n = 2H_{n-1} + 1$ (not homogeneous)

    $B_n = nB_{n-1}$ (coefficients are not constant)

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{k-1} a_{n-(k-1)} + c_k a_{n-k}$

$a_n = r^n$ is a solution iff

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_{k-1} r^{n-(k-1)} + c_k r^{n-k}$$ (r is a constant)

Divide both sides by r^{n-k}

$$\frac{r^n}{r^{n-k}} = c_1 \frac{r^{n-1}}{r^{n-k}} + c_2 \frac{r^{n-2}}{r^{n-k}} + \cdots + c_{k-1} \frac{r^{n-(k-1)}}{r^{n-k}} + c_k \frac{r^{n-k}}{r^{n-k}}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k$$

$$\boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0}$$ (Degree k)

Is called Characteristic equation

The roots are called Characteristic roots

Examples of characteristic equations

Example

$a_n = a_{n-1} + a_{n-2}$

has characteristic equation

$r^2 = r + 1$ (or $r^2 - r - 1 = 0$)

$r^2 - c_1 r - c_2 = 0$ (Degree 2)

Example

$a_\mathrm{n} = 3a_{n-1} + a_{n-5}$

has characteristic equation

$r^5 = 3r^4 + 1$ (or $r^5 - 3r^4 - 1 = 0$)

Theorem

Let c_1 and c_2 be real numbers.

Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$,

where α_1 and α_2 are constants.

Example

Find the solution of $a_n = a_{n-1} + 2a_{n-2}$ for $n > 1$, with initial conditions: $a_0 = 2$ and $a_1 = 7$.

$r^2 - r - 2 = 0$

$(r - 2)(r + 1) = 0$

$r = 2$ or $r = -1$

Solution

characteristic equation: $r^2 = r + 2$ has roots 2, - 1.

Hence $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ for all n for some α_1, α_2.

$\rightarrow a_0 = \alpha_1 + \alpha_2 = 2$

$a_1 = 2\alpha_1 - \alpha_2 = 7 \rightarrow \alpha_1 = 3; \alpha_2 = -1$

$\rightarrow a_n = 3 \times 2^n + (-1) \times (-1)^n$

$= 3 \times 2^n - (-1)^n$ for $n \geq 0$.

Theorem

Let c_1 and c_2 are real numbers with $c_2 \neq 0$.

Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \text{iff}$$

$$a_n = \alpha_1 r_0^{\,n} + \alpha_2 \boldsymbol{n} r_0^{\,n}$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example

find the solution of

$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

Solution

The characteristic equation:

$r^2 = 6r - 9$ has only one root 3.

$r^2 - 6r + 9 = 0$
$(r - 3)(r - 3) = 0$
$r = 3$

$\rightarrow a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

$\rightarrow a_0 = 1 = \alpha_1$, and

$a_1 = 6 = 3\alpha_1 + 3\alpha_2$ $\qquad \alpha_2 = 1$

$\rightarrow a_n = 3^n + n 3^n$ for $n \geq 0$.

Theorem

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$ if and only if

$$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} + \ldots + \alpha_k r_k^{\,n}$$

for $n = 0, 1, 2, \ldots$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example: Higher - Order LHRRWCCs

Solve the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ where $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution

1) Find the general solution of the recurrence relation
   - the characteristic equation is given by $r^3 = 6r^2 - 11r + 6$
   - the characteristic roots are 1, 2 and 3
   - thus, the general solution is $a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$

2) Find the constant values, α_1, α_2, and α_3 using the initial conditions by Solving the linear system:

   $a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 2$, $a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$, $a_2 = \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$

   $(\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2)$

- Thus the unique solution to the recurrence relation and initial conditions is $a_n = 1 - 2^n + 2 \text{ x } 3^n$, $n \geq 0$