2.4 Sequences and Summations

Credit

Mehrdad Nojoumian

Michael P. Frank

Husni Al-Muhtaseb

Sequences

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8, … or 1, 3, 9, 27, 81, …

Definition: a **sequence** is a function from a subset of the integers (usually from {0, 1, 2, 3, …} or {1, 2, 3, …}) to a set S

- The notation $\boldsymbol{a_n}$ is used to denote the image of the integer $\boldsymbol{n}$. We can think of $\boldsymbol{a_n}$ as the equivalent of $f(n)$ where f is a function from {0, 1, 2, …} to S
- The notation $\{a_n\}$ describes the sequence
  - We call $\boldsymbol{a_n}$ a **term** of the sequence

Definition: a **string** is a finite sequence of characters from a finite set (an alphabet)

Geometric Progression

Definition: a **geometric progression** is a sequence of the form:

$\boldsymbol{a, ar, ar^2, \ldots, ar^n, \ldots}$

where the initial term $\boldsymbol{a}$ and the common ratio $\boldsymbol{r}$ are real numbers.

Geometric Progression

Definition: a **geometric progression** is a sequence of the form: $\boldsymbol{a, ar, ar^2, \ldots, ar^n, \ldots}$ where the initial term $\boldsymbol{a}$ and the common ratio $\boldsymbol{r}$ are real numbers.

Example:

Consider the sequence $\{b_n\} = b_0, b_1, \ldots$
where $b_n = (-1)^n$

$\{b_n\} = b_0, b_1, b_2, b_3, b_4, \ldots = 1, -1, 1, -1, 1, \ldots$

$a = 1$ and $r = -1$

Geometric Progression

Definition: a **geometric progression** is a sequence of the form: $\boldsymbol{a, ar, ar^2, \ldots, ar^n, \ldots}$ where the initial term $\boldsymbol{a}$ and the common ratio $\boldsymbol{r}$ are real numbers.

Example:

Consider the sequence $\{c_n\} = c_0, c_1, \ldots$
where $c_n = 2 \times 5^n$

$$\{c_n\} = c_0, c_1, c_2, c_3, c_4, \ldots = 2, 10, 50, 250, 1250, \ldots$$

$a = 2$ and $r = 5$

Geometric Progression

Definition: a **geometric progression** is a sequence of the form: $\boldsymbol{a, ar, ar^2, \ldots, ar^n, \ldots}$ where the initial term $\boldsymbol{a}$ and the common ratio $\boldsymbol{r}$ are real numbers.

Example:

Consider the sequence $\{d_n\} = d_0, d_1, \ldots$

where $d_n = 6 \times \left(\frac{1}{3}\right)^n$

$\{d_n\} = d_0, d_1, d_2, d_3, d_4, \ldots = 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

$a = 6$ and $r = \frac{1}{3}$

Arithmetic Progression

Definition: an ***arithmetic progression*** is a sequence of the form:

$\boldsymbol{a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots}$

where the initial term $\boldsymbol{a}$ and the common difference $\boldsymbol{d}$ are real numbers.

Arithmetic Progression

Definition: an ***arithmetic progression*** is a sequence of the form:
a*, *a* + *d*, *a* + 2*d*, *a* + 3*d*, ..., *a* + *nd*, ...** where the initial term ***a and the common difference ***d*** are real numbers

Example:

The sequences $\{s_n\}$ with $s_n = -1 + 4n$

$\{s_n\} = s_0, s_1, s_2, s_3, s_4, \ldots = -1, 3, 7, 11, 15, \ldots$

$a = -1$ and $d = 4$

Arithmetic Progression

Definition: an ***arithmetic progression*** is a sequence of the form:
a*, *a* + *d*, *a* + 2*d*, *a* + 3*d*, ..., *a* + *nd*, ...** where the initial term ***a and the common difference ***d*** are real numbers

Example:

The sequences $\{t_n\}$ with $t_n = 7 - 3n$

$\{t_n\} = t_0, t_1, t_2, t_3, t_4, \ldots = 7, 4, 1, -2, -5, \ldots$

$$a = 7 \text{ and } d = -3$$

Arithmetic Progression

Definition: an ***arithmetic progression*** is a sequence of the form:

a*, *a* + *d*, *a* + 2*d*, *a* + 3*d*, …, *a* + *nd*, …** where the initial term ***a and the common difference ***d*** are real numbers

Example:

The sequences $\{u_n\}$ with $u_n = 1 + 2n$

$\{u_n\} = u_0, u_1, u_2, u_3, u_4, \ldots = 1, 3, 5, 7, 9, \ldots$

$a = 1$ and $d = 2$

Recurrence Relations

Definition: a **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the **previous terms of the sequence**:

Example: $a_n = a_{n-1} + 3$

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation
- The **initial conditions** for a sequence specify the terms that precede the first term where the recurrence relation takes effect

Recurrence Relations

Example: let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + \mathbf{3} \quad \text{for} \quad n = 1, 2, 3, 4, \ldots$$

suppose that $a_0 = 2$, what are a_1, a_2 and a_3?

Solution: here $\boldsymbol{a_0 = 2}$ is the initial condition. We see from the recurrence relation that

$a_1 = \mathbf{2} + \mathbf{3} = \mathbf{5}$

$a_2 = \mathbf{5} + \mathbf{3} = \mathbf{8}$

$a_3 = \mathbf{8} + \mathbf{3} = 11 \ldots$

Recurrence Relations

Example: let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} - a_{n-2} \quad \text{for} \quad n = 2, 3, 4, \ldots$$

suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: The initial conditions are $\boldsymbol{a_0 = 3}$ and $\boldsymbol{a_1 = 5}$

From the recurrence relation

$$a_2 = a_1 - a_0 = \mathbf{5} - \mathbf{3} = \mathbf{2}$$

$$a_3 = a_2 - a_1 = \mathbf{2} - \mathbf{5} = \mathbf{-3}$$

$$a_4 = a_3 - a_2 = \mathbf{-3} - \mathbf{2} = -5 \ldots$$

Fibonacci Sequence

Definition: define the **Fibonacci sequence**, $f_0, f_1, f_2, \ldots$ by:

- Initial conditions: $f_0 = 0, f_1 = 1$
- Recurrence relation: $f_n = f_{n-1} + f_{n-2}$

Example: find f_2, f_3, f_4, f_5 and f_6

Solution:

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$

Solving Recurrence Relations

- Finding a formula for the n^{th} term of the sequence, generated by a recurrence relation, is called **solving** the recurrence relation. Such a formula is called a **closed formula**.

Iterative Solution Example

Start from the first term and substitute forward to term n

Method 1: forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

$$a_2 = a_1 + 3 = 2 + 3 = 2 + \mathbf{1} \cdot 3$$

$$a_3 = a_2 + 3 = (2 + 1 \cdot 3) + 3 = 2 + \mathbf{2} \cdot 3$$

$$a_4 = a_3 + 3 = (2 + 2 \cdot 3) + 3 = 2 + \mathbf{3} \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = 2 + \mathbf{(n - 1)} \cdot 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Start from term n and substitute back until the initial term

Method 2: **backward substitution**

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 = a_{n-1} + 1 \cdot 3 \\
&= (a_{n-2} + 3) + 1 \cdot 3 = a_{n-2} + 2 \cdot 3 \\
&= (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3 \\
&\;\;\vdots \\
&= a_{n-(n-1)} + (n-1) \cdot 3 \\
&= a_1 + (n-1) \cdot 3 \\
&= 2 + (n-1) \cdot 3 == 2 + 3(n-1)
\end{aligned}
$$

Financial Application

Example: suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually.

How much will be in the account after 30 years?

Solution: let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$\boldsymbol{P_n = P_{n\text{-}1} + 0.11\,P_{n\text{-}1} = (1.11)\,P_{n\text{-}1}}$$

with the initial condition $\boldsymbol{P_0 = 10{,}000}$

Financial Application

$P_n = (1.11)\, P_{n-1}$ by forward substitution, we get:

$P_1 = (1.11)\, P_0$

$P_2 = (1.11)\, P_1 = (1.11) \cdot (1.11)\, P_0 = (1.11)^2\, P_0$

$P_3 = (1.11)\, P_2 = (1.11) \cdot (1.11)^2\, P_0 = (1.11)^3\, P_0$

$\vdots$

$P_n = (1.11) P_{n-1} = (1.11)^n\, \mathrm{P}_0$

$P_n = (1.11)^n\, 10{,}000$

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Special Integer Sequences

- Given a few terms of a sequence, try to identify the sequence, i.e., obtain a formula or a recurrence relation.
- **Some questions to ask?**
  - Are there repeated terms of the **same value**?
  - Can you obtain a term from the previous term by **adding** an amount or **multiplying** by an amount?
  - Can you obtain a term by **combining** the previous terms in some way?
  - Are there **cycles** among the terms?
  - Do the terms match those of a **well known sequence**?

Special Integer Sequences

Example: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution: Note that the denominators are powers of 2.

The sequence with $\boldsymbol{a_n = \frac{1}{2^n}}$ is a possible match

This is a geometric progression with

$$a = 1 \text{ and } r = \frac{1}{2}$$

Special Integer Sequences

Example: **1, 3, 5, 7, 9, …**

Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is

$$a_n = 2n + 1.$$

This is an arithmetic progression with

$$a = 1 \text{ and } d = 2$$

Special Integer Sequences

Example: **1, -1, 1, -1, 1, …**

Solution: The terms alternate between 1 and -1.

A possible sequence is

$$a_n = (-1)^n$$

This is a geometric progression with

$a = 1$ and $r = -1$

Useful Sequences

n^{th} Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …*

* (Fibonacci sequence with different initial conditions)

Guessing Sequences

Example: conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are

1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: note the ratio of each term to the previous approximates 3.

So now compare with the sequence $\mathbf{3^n}$.

3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...

We notice that the n^{th} term is **2** less than the corresponding power of 3.

So a good conjecture is that

$$\boldsymbol{a_n = 3^n - 2}$$

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$

$$a_m + a_{m+1} + \ldots + a_n$$

- The notation:

$$\sum_{j=m}^{n} a_j$$

$$\sum\nolimits_{j=m}^{n} a_j \, , \qquad or \; \sum\nolimits_{m \le j \le n} a_j$$

The variable ***j*** is called the ***index of summation***.

$$\sum\nolimits_{j=m}^{n} a_j \quad = \quad \sum\nolimits_{i=m}^{n} a_i \quad = \quad \sum\nolimits_{k=m}^{n} a_k$$

It runs through all the integers starting with its ***lower limit "m"*** and ending with its ***upper limit "n"***.

Summations

- More generally for a set S: $\sum_{j \in S} a_j$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{j=0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

If S = {2, 5, 7, 10} then

$$\sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Summations

- We can also use summation notation to add all values of a function, where the index of summation runs over all values in a set.

$$\sum_{j \in S} f(j)$$

If $S = \{2, 5, 7, 10\}$ then

$$\sum_{j \in S} f(j) = f(2) + f(5) + f(7) + f(10)$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$

$$a_m \times a_{m+1} \times \ldots \times a_n$$

The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod\nolimits_{j=m}^{n} a_j \qquad \prod\nolimits_{m \le j \le n} a_j$$

Some Useful Summation Formulae

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \ \|x\| < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \ \|x\| < 1$	$\frac{1}{(1 - x)^2}$

Simple Summation Example

$$\sum_{i=2}^{4}(i^2+1)$$
$$= (2^2+1)+(3^2+1)+(4^2+1)$$
$$= (4+1)+(9+1)+(16+1)$$
$$= 5+10+17$$
$$= 32$$

Examples

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} \quad = 2^0 + 2^{-1} + 2^{-2} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \land \\ x<10}} x^2 \; = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87$$

Summation Manipulations

- Some identities for summations:

$$\sum_x cf(x) = c \sum_x f(x) \qquad c \text{ is a constant}$$

$$\sum_x (f(x) + g(x)) = \left(\sum_x f(x)\right) + \sum_x g(x)$$

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad where\ j \le m < k$$

Example

Evaluate $\sum_{k=50}^{100} k^2$

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

Use series splitting

Solve for desired summation

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Apply quadratic series rule

$$= \frac{100 \times 101 \times 201}{6} - \frac{49 \times 50 \times 99}{6}$$

Evaluate

$$= 338350 - 40425 = 297925$$

Nested Summations

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

$$= \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right)$$

$$= \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right)$$

$$= \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6 \,.\, 10 = 60$$