1.5 Nested Quantifiers

Credit

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1.5 Nested quantifiers

Assume the domain for the variables x, and y consists of real numbers R

$\forall x\, \exists y\, (x + y) = 0$ says:

For every real number x, there exists a real number y where $x + y = 0$

(this is true if we let y be $-x$)

$\forall x\, \exists y\, (x + y) = 0$ same as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ and $P(x, y)$ is $(x + y) = 0$

$\forall x\, \forall y\, (x + y) = (y + x)$ says:

For every real number x, and every real number y,

$(x + y) = (y + x)$

- *commutative law for addition of real numbers*

Nested quantifiers

Similarly, the *associative law:*

$\forall x \, \forall y \, \forall z \;\; (x + (y + z)) = ((x + y) + z)$

Example: the domain is R

$\forall x \, \forall y \, ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$ says ?

The multiplication of two negative real numbers is a positive real number

Quantification as loop

$$\forall x\ \forall y\ P(x, y)$$

$$\forall x\ \exists y\ P(x, y)$$

$$\exists x\ \forall y\ P(x, y)$$

$$\exists x\ \exists y\ P(x, y)$$

Quantification as loop

- For every x, for every y $\forall x \, \forall y \, P(x, y)$
  - Loop through x and for each x loop through y
  - If we find $P(x, y)$ is true for all x and y, then the statement is true
  - If we ever hit a value x for which we hit a value for which $P(x, y)$ is false, the whole statement is false
- For every x, there exists y $\forall x \, \exists y \, P(x, y)$
  - Loop through x until we find a y that $P(x, y)$ is true
  - If for every x, we find such a y, then the statement is true

Quantification as loop

- $\exists x \, \forall y \, P(x, y)$: we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y. Once we find such an x, then it is true.
- $\exists x \, \exists y \, P(x, y)$: loop though the values for x where for each x loop through the values of y until we find an x for which we find a y such that $P(x, y)$ is true.
  - False only if we never hit an x for which we never find y such that $P(x, y)$ is true

Order of quantification

Order is important unless all quantifiers are universal quantifiers or all are existential quantifiers

Let $P(x, y)$ be the statement "$x + y = y + x$", domain is R.

What are the truth values of

$\forall x \, \forall y \, P(x, y)$

$\forall y \, \forall x \, P(x, y)$

Order of quantification

$\forall x\, \forall y\, P(x, y)$ means

"for all real numbers x, for all real numbers y, $x + y = y + x$."

an axiom for the real numbers R (Appendix 1)

$\forall x\, \forall y\, P(x, y)$ is true

$\forall y\, \forall x\, P(x, y)$ means

"for all real numbers y, for all real numbers x, $x + y = y + x$."

same meaning as the above statement

$\forall y\, \forall x\, P(x, y)$ is true

The order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement. $\forall x\, \forall y\, P(x, y) \equiv \forall y\, \forall x\, P(x, y)$

Order of quantification

Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y\, \forall x\, Q(x, y)$ and $\forall x\, \exists y\, Q(x, y)$, Domain is R.

$\exists y\, \forall x\, Q(x, y)$ denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y\, \forall x\, Q(x, y)$ is false.

$\forall x\, \exists y\, Q(x, y)$ denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, $\forall x\, \exists y\, Q(x, y)$ is true.

Here the order in which quantifiers appear makes a difference. Be careful with the order of existential and universal quantifiers!

Quantification of two variables

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Try this at home

Order of Quantifiers

- Example: Let $Q(x, y, z)$ be the statement "$x + y = z$". What are the truth values of the quantifications $\forall x\, \forall y\, \exists z\, Q(x, y, z)$ and $\exists z\, \forall x\, \forall y\, Q(x, y, z)$, where the domain for all variables consists of all real numbers?

- Sol: $\forall x\, \forall y\, \exists z\, Q(x, y, z)$ = "For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$" TRUE

  The order of quantification is important here. Since $\exists z\, \forall x\, \forall y\, Q(x, y, z)$ = "There is a real number z such that for all real numbers x and for all real numbers y, $x + y = z$" FALSE

It is false because there is no z that satisfies the equation $x + y = z$ for all real numbers x and y

Translating mathematical statements with Nested Quantifiers

- "The sum of two positive integers is always positive"
- First: we rewrite it so that the implied quantifiers and a domain are shown:
  - "For every two integers, if these integers are both positive, then the sum of these integers is positive."
- Next: we introduce the variables x and y to obtain
  - "For all positive integers x and y, $x + y$ is positive."
- Next: $\forall x \, \forall y \, ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$
  - where the domain for both variables consists of all integers.

Translating mathematical statements with Nested Quantifiers

- Note that we could also translate this using the positive integers as the domain
- “The sum of two positive integers is always positive”
- “For every two positive integers, the sum of these integers is positive”
- $\forall x\, \forall y\, (x + y > 0)$
  - where the domain for both variables consists of all positive integers.

Example

- "Every real number except zero has a multiplicative inverse"

(A **multiplicative inverse** of a real number x is a real number y such that $xy = 1$)

- "For every real number x except zero, x has a multiplicative inverse."
- "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$."
- $\forall x\,((x \neq 0) \rightarrow \exists y\,(xy = 1))$
- Where the domain for both variables consists of real numbers

Translating predicates with nested Quantifiers to English

- Assume $C(x)$ is "x owns a computer", $F(x, y)$ is "x and y are friends", and the domain for x and y consists of all students in KFUPM.
  - Express the following in English:
- $\forall x\,(C(x) \vee \exists y\,(C(y) \wedge F(x, y)))$
  - Answer: *Every KFUPM student owns a computer or is a friend of a KFUPM student who owns a computer*.

Translating predicates to English

- Assume $F(x, y)$ means x and y are friends, and the domain consists of all students in KFUPM
  - Express the following in English:
- $\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y, z))$
- There is a student none of whose friends are also friends with each other.

Do you want to try this at home?

Negating nested quantifiers

$$\neg\forall x\,\exists y\,(xy = 1)$$
$$\equiv \exists x\,\neg\exists y\,(xy = 1)$$
$$\equiv \exists x\,\forall y\,\neg(xy = 1)$$
$$\equiv \exists x\,\forall y\,(xy \neq 1).$$

Negating nested quantifiers

- Use quantifiers to express:
  - There does not exist a woman who has taken a flight on every airline in the world.
- Sol:
  - This statement is the negation of the statement “There is a woman (w) who has taken a flight (f) on every airline (a) in the world”
- Let $P(w, f)$ is “w has taken f”, and $Q(f, a)$ is “f is a flight on a”
  - $\neg\exists w\, \forall a\, \exists f\, (P(w, f) \wedge Q(f, a))$