Predicates and Quantifiers

Credit

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1.4 Predicate and quantifiers

- Can be used to express the meaning of a wide range of statements
- Allow us to reason and explore relationship between objects
- **Predicates**: statements involving variables e.g.
  - "$x > 3$"
  - "$x = y + 3$"
  - "$x + y = z$"
  - "computer x is under attack by an intruder"
  - "computer y is functioning properly"

Example: $x > 3$

- The variable x is the subject of the statement
- **Predicate** "is greater than 3" refers to a property that the subject of the statement can have
- Can denote the statement by $P(x)$
  - where P denotes the predicate "is greater than 3" and x is the variable
- $P(x)$: also called the value of the **propositional function** P at x
- Once a value is *assigned* to the variable x, $P(x)$ becomes a proposition and has a truth value

Example

- Let $P(x)$ denote the statement "$x > 3$"
  - $P(4)$: setting $x = 4$, $4 > 3$ is true, thus $P(4)$ is TRUE
  - $P(2)$: setting $x = 2$, $2 > 3$ is false, thus $P(2)$ is FALSE
- Let $A(x)$ denote the statement "computer x is under attack by an intruder".
- Suppose that only Computers with the names CS2 and MATH1 are currently under attack
  - $A(\mathrm{CS1})$? : FALSE
  - $A(\mathrm{CS2})$? : TRUE
  - $A(\mathrm{MATH1})$?: TRUE

  - CS1, CS2, and MATH1 are computers' names

n-ary Predicate

- A statement involving n variables, $x_1, x_2, \ldots, x_n$, can be denoted by $P(x_1, x_2, \ldots, x_n)$
- $P(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$
- P is also called ***n-ary*** **predicate**

Universe of Discourse (domain)

- Consider the statement '$x > 3$',
  - does it make sense to assign to x the value 'blue'?
- Intuitively, the **universe of discourse** (domain) is
  - the set of all things we wish to talk about;
  - that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

  Enrolled_ICS253(x) = 'x is enrolled in ICS253'
- The collection of values that a variable x can take is called x's universe of discourse.

Quantifiers (∀, ∃)

- Express the extent to which a predicate is TRUE
  - In English, *all*, *some*, *many*, *none*, *few*
- Focus on two types: (∀, ∃)
  - **Universal**: a predicate is true for **every** element under consideration ∀
  - **Existential**: a predicate is true for **one or more** elements under consideration ∃
- **Predicate calculus**: the area of logic that deals with predicates and quantifiers

Universal quantifier ($\forall$)

- "$P(x)$ for all values of x in the domain"

$$\forall x\, P(x)$$

- Read it as "for all x $P(x)$" or "for every x $P(x)$"
- A statement $\forall x\, P(x)$ is false if and only if $P(x)$ is not always true
- An element for which $P(x)$ is false is called a **counterexample** of $\forall x\, P(x)$
- A single counterexample is enough to establish that $\forall x\, P(x)$ is not true

Example

- Let $P(x)$ be the statement "$x + 1 > x$".
  - What is the truth value of $\forall x\, P(x)$?
    - Implicitly assume the domain of a predicate is not empty
  - Because $P(x)$ is true for all real numbers x, the quantification
    - $\forall x\, P(x)$ is true
- Let $Q(x)$ be the statement "$x < 2$".
  - What is the truth value of $\forall x\, Q(x)$ where the domain consists of all real numbers?
  - $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false.
    - That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$.
    - Thus $\forall x\, Q(x)$ is false.

Example

- Let $P(x)$ be "$x^2 > 0$". To show that the statement $\forall x\, P(x)$ is false where the domain consists of all integers
  - Show a counterexample
    - with $x = 0$
- When all the elements can be listed, e.g.,

  $x_1, x_2, \ldots, x_n$,

  it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

Example

- What is the truth value of $\forall x\, P(x)$ where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of positive integers not exceeding 4?

$\forall x\, P(x)$ is the same as

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$$

As $P(4) = 16$ is false, $\forall x\, P(x)$ is false

$P(4)$ *is a counterexample*

Existential quantification (∃)

- "There exists an element x in the domain such that $P(x)$ (is true)"
- Denoted as $\exists x\ P(x)$ where $\exists$ is the existential quantifier
- In English, "for some", "for at least one", or "there is"
- Read as:
  - "There exists an x such that $P(x)$",
  - "There is an x such that $P(x)$",
  - "There is at least one x such that $P(x)$", or
  - "For some x, $P(x)$"

Example

- Let $P(x)$ be the statement "$x > 3$".
  - Is $\exists x\, P(x)$ true for the domain of all real numbers? True
- Let $Q(x)$ be the statement "$x = x + 1$".
  - Is $\exists x\, Q(x)$ true for the domain of all real numbers? False
- When all elements of the domain can be listed, e.g.,

  $x_1, x_2, \ldots, x_n,$

  it follows that the existential quantification is the same as disjunction

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Example

- What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the domain consists of positive integers not exceeding 4?

$\exists x\, P(x)$ is the same as

$$P(1) \vee P(2) \vee P(3) \vee P(4)$$

$P(1)$ is $1^2 > 10$ False

$P(2)$ is $2^2 > 10$ False

$P(3)$ is $3^2 > 10$ False

$P(4)$ is $4^2 > 10$ **True**

Thus, $\exists x\, P(x)$ is True

Actually, $P(4)$ is $4^2 > 10$ **True**

Uniqueness quantifier ($\exists!$ $\exists_1$)

- There exists a unique x such that $P(x)$ is true

$$\exists! x\, P(x)$$

- “There is exactly one”, “There is one and only one”

$$\exists_1 x\, P(x)$$

Quantifiers with restricted domains

- What do the following statements mean for the domain of real numbers?
  - $\forall x < 0\ (x^2 > 0)$
    - The square of a negative real number is positive.
    - same as $\forall x\ (x < 0 \rightarrow x^2 > 0)$
  - $\forall y \neq 0\ (y^3 \neq 0)$
    - The cube of every nonzero real number is nonzero.
    - same as $\forall y\ (y \neq 0 \rightarrow y^3 \neq 0)$
  - $\exists z > 0\ (z^2 = 2)$
    - There is a positive square root of 2.
    - same as $\exists z\ (z > 0 \land z^2 = 2)$

Be careful about $\rightarrow$ and $\land$ in these statements

Quantification Examples

$P(x)$ = "$x + 1 = 2$"

Domain is R (set of real numbers)

Proposition	Truth Value
$\forall x\ P(x)$	F
$\forall x\ \neg P(x)$	F
$\exists x\ P(x)$	T
$\exists x\ \neg P(x)$	T
$\exists! x\ P(x)$	T
$\exists! x\ \neg P(x)$	F

Quantification Examples

- $P(x) =$ "$x^2 > 0$"

Domain	Proposition	Truth Value
R	$\forall x\ P(x)$	F
Z	$\forall x\ P(x)$	F
Z - {0}	$\forall x\ P(x)$	T
Z	$\exists! x\ \neg P(x)$	T
N = {1,2, ..}	$\exists x\ \neg P(x)$	F

Quantification Examples

Proposition	Truth Value	
$\forall x \in \mathrm{R} \quad (x^2 \geq x)$	F	Check a numbers in (0, 1)
$\exists! x \in \mathrm{R} \quad (x^2 < x)$	F	All numbers in (0, 1)
$\forall x \in (0,1) \ (x^2 < x)$	T	
$\forall x \in \{0,1\} \ (x^2 = x)$	T	
$\forall x \in \varnothing \ P(x)$	T	

if the domain is empty, then $\forall x\, P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Precedence of quantifiers

- $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus

$\forall x \; P(x) \lor Q(x)$ is equivalent to $(\forall x \; P(x)) \lor Q(x)$

Rather than $\forall x \, (P(x) \lor Q(x))$

Binding variables

- When a quantifier is used on the variable x, this occurrence of variable is **bound**
- If a variable is not bound, then it is **free**
- All variables occur in propositional function of predicate calculus must be bound or set to a particular value to turn it into a proposition
- The part of a logical expression to which a quantifier is applied is the **scope** of this quantifier

Example

What are the scopes of these expressions?
Are all the variables bound?

$\exists x\,(x + y = 1)$

x is bound, but y is free

$\exists x\,(P(x) \wedge Q(x)) \vee \forall x\, R(x)$

All variables are bound
Scope of the first x is $(P(x) \wedge Q(x))$
Scope of the second x is $R(x)$

Same as:
$\exists x\,(P(x) \wedge Q(x)) \vee \forall y\, R(y)$

The same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap

Logical equivalences

- $S \equiv T$: Two statements S and T involving predicates and quantifiers are logically equivalent *If* and *only if*
  - they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables.
  - Example: $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

  i.e., we can distribute a universal quantifier over a conjunction
  - Example: $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

  i.e., we can distribute an existential quantifier over a disjunction

$$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

- Both statements must take the same truth value no matter the predicates P and Q, and no matter which domain is used
- Show
  - If LHS is true, then RHS is true (LHS → RHS)
  - If RHS is true, then LHS is true (RHS → LHS)

$$\forall x\,(P(x) \wedge Q(x)) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$$

$$\forall x\,(P(x) \wedge Q(x)) \leftarrow \forall x\, P(x) \wedge \forall x\, Q(x)$$

$$\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\,(P(x) \wedge Q(x))$$

Negating Quantified Expressions

- $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
  - Negation of the statement "Every student in the class has taken a course in Calculus"
  - "It is not the case that every student in the class has taken a course in Calculus."
  - This is equivalent to
  - "There is a student in the class who has not taken a course in Calculus"

Negating Quantified Expressions

- $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$
  - Negation of the statement “There is a student in this class who has taken a course in calculus.”
  - “It is not the case that there is a student in this class who has taken a course in calculus.”
  - This is equivalent to “Every student in this class has not taken calculus”

Negating quantified expressions

Negation: $\neg\exists\, xP(x)$
Equivalent Statement: $\forall x\ \neg P(x)$
When Is Negation True: For every x, $P(x)$ is false.
When False: There is an x for which $P(x)$ is true.

Negation: $\neg\forall x\ P(x)$
Equivalent Statement: $\exists x\ \neg P(x)$
When Is Negation True: There is an x for which $P(x)$ is false.
When False: $P(x)$ is true for every x.

De Morgan's Laws for Quantifiers.

Negate the following statement
"There is an honest politician"
"Every politician is dishonest"

Examples

What is the negations of the statement $\forall x\,(x^2 > x)$?

Solution:

$\neg\forall x\,(x^2 > x)$

$\equiv \exists x\,\neg(x^2 > x)$

$\equiv \exists x\,(x^2 \le x)$

What is the negations of the statement $\exists x\,(x^2 = 2)$?

Solution:

$\neg\,\exists x\,(x^2 = 2)$

$\equiv \forall x\,\neg(x^2 = 2)$

$\equiv \forall x\,(x^2 \ne 2)$

Show that $\neg \forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \land \neg Q(x))$

Solution:

$\neg \forall x\,(P(x) \rightarrow Q(x))$

$\equiv \exists x\,(\neg(P(x) \rightarrow Q(x)))$ De Morgan's Law

$\equiv \exists x\,(\neg(\neg P(x) \lor Q(x)))$ Implication Equivalence

$\equiv \exists x\,(\neg(\neg P(x)) \land \neg Q(x))$ De Morgan's Law

$\equiv \exists x\,(P(x) \land \neg Q(x))$ Double Negation

Translating English into logical expressions

- "Every student in this class has studied calculus"

Let $C(x)$ be "x has studied calculus"

Let $S(x)$ be "x is in this class"

If the domain consists of students of this class

$$\forall x\, C(x)$$

If the domain consists of all people

$$\forall x (S(x) \rightarrow C(x))$$

What about:

$$\forall x (S(x) \land C(x))$$ ✗ **Why?**

This statement says that all people are students in this class and have studied calculus

Using quantifiers in system specifications

- "Every mail message larger than one megabyte will be compressed"

Let $S(m, y)$ be "mail message m is larger than y megabytes"

where m has the domain of all mail messages and y is a positive real number.

Let $C(m)$ denote "message m will be compressed"

$$\forall m\ (S(m, 1) \rightarrow C(m))$$

Example

- "If a user is active, at least one network link will be available"
  - Let $A(u)$ represent "user u is active" where u has the domain of all users
  - and let $S(n, x)$ denote "network link n is in state x" where
    - n has the domain of all network links, and
    - x has the domain of all possible states, {available, unavailable}.

$$\exists u\, A(u) \rightarrow \exists n\, S(n, \text{available})$$

Do you want to try this at home?