

2.5 Cardinality of Sets

Credit

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Cardinality

Definition: the **cardinality** of a set A is equal to the cardinality of a set B , denoted by $|A| = |B|$ if and only if there is a one-to-one correspondence (bijection) from A to B .

- If there is a one-to-one function (injection) from A to B , the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

Definition: a set that is either finite or has the same cardinality as the set of positive integers ($\mathbb{Z}^+ : \{1, 2, \dots\}$) is called **countable**. A set that is not countable is **uncountable** (e.g., Real numbers).

Showing that a Set is Countable

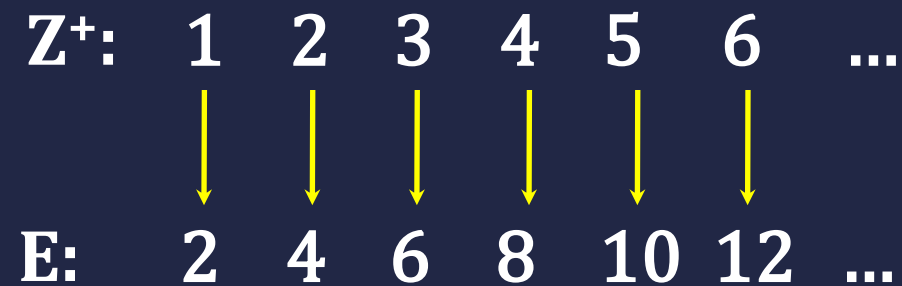
- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence indexed by the positive integers \mathbf{Z}^+ .
- The reason for this is that a one-to-one correspondence f (bijection) from the set of positive integers \mathbf{Z}^+ to a set S can be expressed in terms of a sequence as follows:

$$a_1, a_2, \dots, a_n, \dots \text{ where } a_1 = f(1), a_2 = f(2), \dots, a_n = f(n), \dots$$

Showing that a Set is Countable

Example: show that the set of positive even integers E is a countable set.

Solution: let $f(x) = 2x$.



Then f is a **bijection** from $\mathbf{Z^+}$ to \mathbf{E} since f is both one-to-one and onto.

Positive Rational Numbers are Countable

- **Reminder:** a rational number can be expressed as the ratio of two integers p and q $\left(\frac{p}{q}\right)$ such that $q \neq 0$.
 - $\frac{3}{4}$ is a rational number
 - $\sqrt{2}$ is irrational (not rational)

Example: show that the positive rational numbers are countable.

Solution: the positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \dots$$

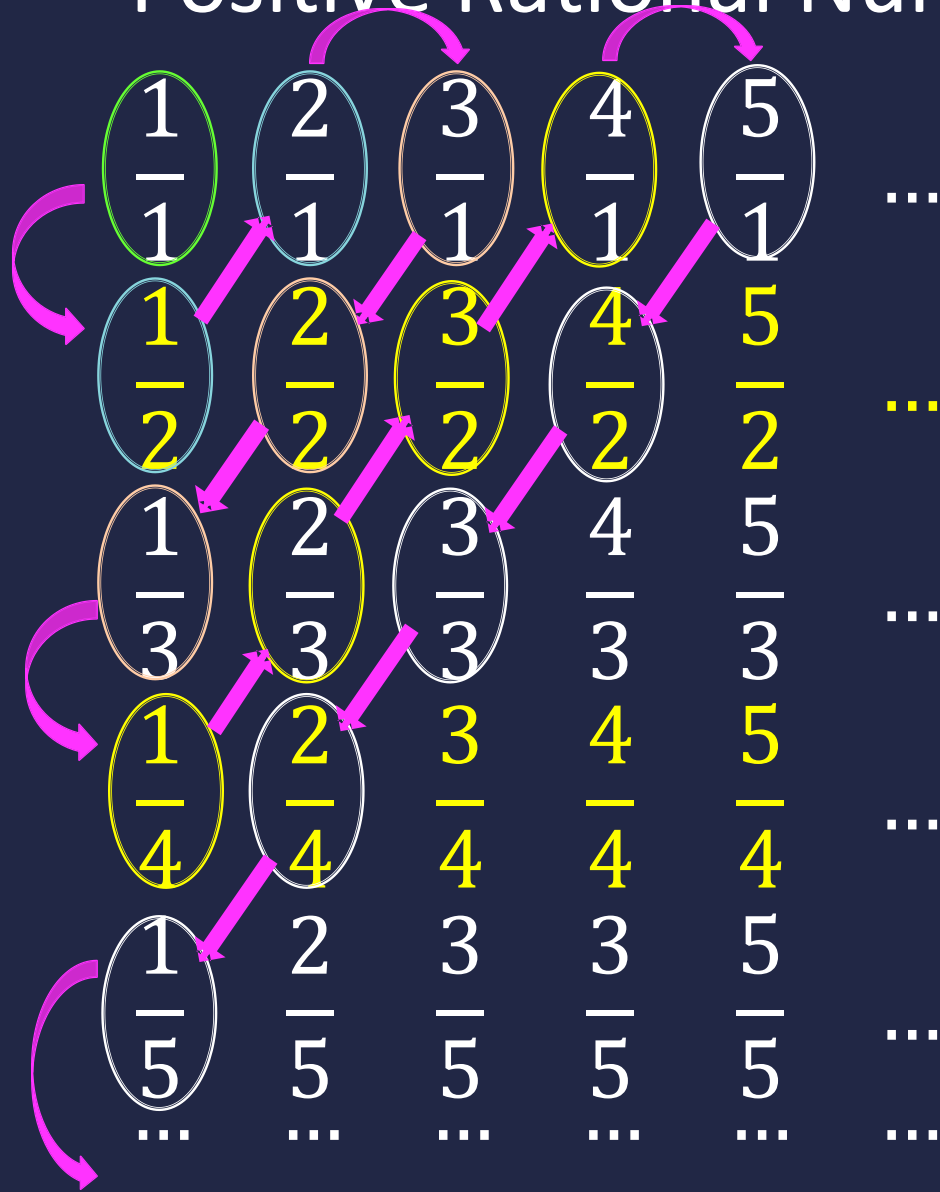
The next slide shows how this is done.



Positive Rational Numbers are Countable $\frac{p}{q}$

First row $q = 1$.	→	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$...
Second row $q = 2$	→	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$...
etc.		$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$...
		$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$...
Fifth row $q = 5$	→	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{5}{5}$...
	

Positive Rational Numbers are Countable $\frac{p}{q}$



Constructing the List

First list $\frac{p}{q}$ with $p + q = 2$

Second list $\frac{p}{q}$ with $p + q = 3$

Third list $\frac{p}{q}$ with $p + q = 4$, and so on.

Terms not circled are not listed because they were previously listed items

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$...
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$...
⋮	⋮								

Countable Sets

- A set is countable if it is finite or if it can be placed in a 1-1 correspondence with the set of natural numbers,
 $\mathbb{N} = \{1, 2, 3, \dots\}$.
- A countable set that is infinite has a cardinality of *aleph-null*. The symbol for *aleph-null* is $\aleph_0 \equiv |\mathbb{N}|$

- The set $\mathbf{R}_{[0,1]}$ of reals between 0 and 1 is not countable.
- If A and B are countable sets, then $A \cup B$ is also countable.
- Any subset of a countable set is also countable

A Computable Function

- A function is *computable* if there is a computer program in some programming language that finds the values of this function
 - even with unlimited time and memory
- If a function is not computable we say it is *uncomputable*.