2.2 Set Operations

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Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*.
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set *U*. **All sets are assumed to be subsets of *U*.**

Union

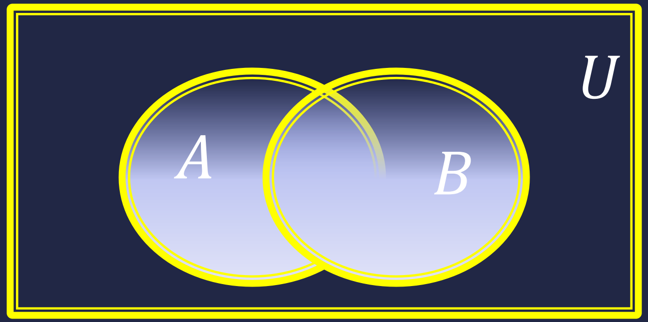
- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

- **Example**: What is $\{1, 2, 3\} \cup \{3, 4, 5\}$?

Solution: $\{1, 2, 3, 4, 5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x \mid x \in A \wedge x \in B\}$$

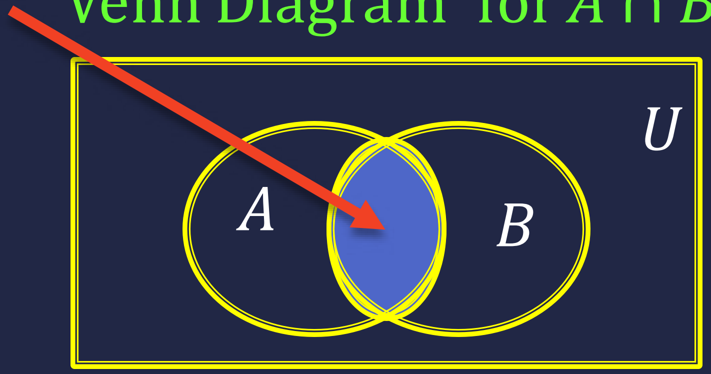
- Note if the intersection is empty, then A and B are said to be ***disjoint***.
- **Example**: What is? $\{1, 2, 3\} \cap \{3, 4, 5\}$?

  Solution: $\{3\}$
- **Example:** What is?

  $\{1, 2, 3\} \cap \{4, 5, 6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

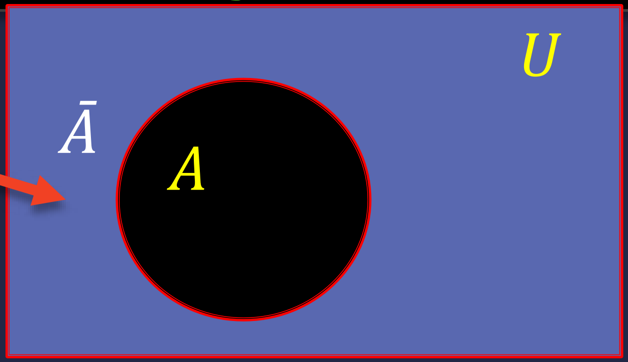
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

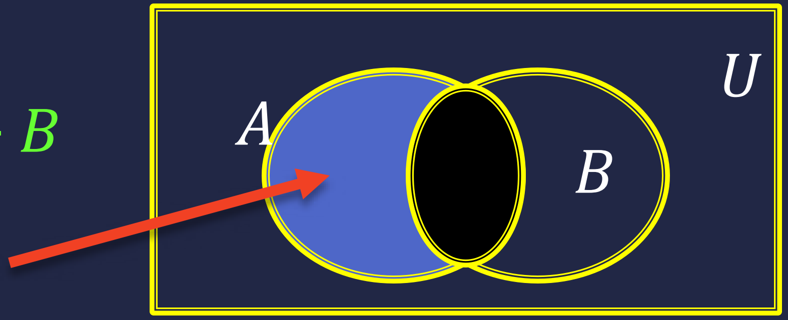


Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

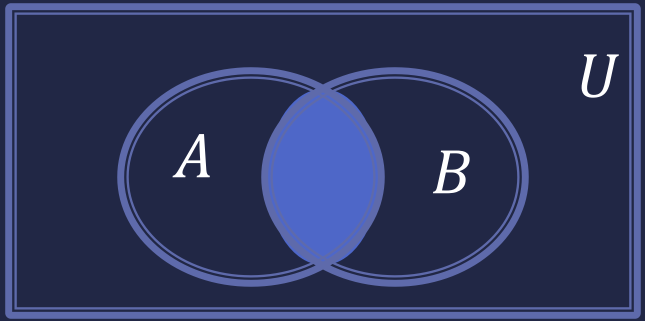
Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

- Inclusion - Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the EE majors in your class and B be the CS majors. To count the number of students who are either EE majors or CS majors, add the number of EE majors and the number of CS majors, and subtract the number of double CS/EE majors.

Review Questions

Example: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$

$A \cup B$

Solution: $\{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cap B$

Solution: $\{4, 5\}$

$\bar{A}$

Solution: $\{0, 6, 7, 8, 9, 10\}$

$\bar{B}$

Solution: $\{0, 1, 2, 3, 9, 10\}$

$A - B$

Solution: $\{1, 2, 3\}$

$B - A$

Solution: $\{6, 7, 8\}$

Symmetric Difference

Definition: The *symmetric difference* of A and B, denoted by

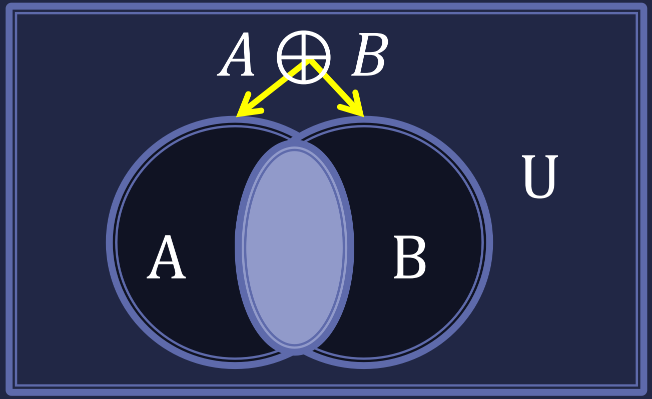
$A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

Example:

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{1, 2, 3, 4, 5\}$
$B = \{4, 5, 6, 7, 8\}$
What is $A \oplus B$

Solution: $\{1, 2, 3, 6, 7, 8\}$



Venn Diagram

Set Identities

- Identity laws

  $A \cap U = A$ $\quad$ $A \cup \emptyset = A$

- Domination laws

  $A \cup U = U$ $\quad$ $A \cap \emptyset = \emptyset$

- Idempotent laws

  $A \cup A = A$ $\quad$ $A \cap A = A$

- Complementation law

  $\overline{(\bar{A})} = A$

(Left to right and Right to left)

$A \cap U = A$ $\quad$ $A \cup \emptyset = A$

$U = A \cup U$ $\quad$ $\emptyset = A \cap \emptyset$

$A = A \cup A$ $\quad$ $A = A \cap A$

$A = \overline{(\bar{A})}$

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Set Identities

- Commutative laws

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(Left to right and Right to left)

Commutative laws

$$B \cup A = A \cup B \quad B \cap A = A \cap B$$

Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive laws

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$$
$$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$$

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Set Identities

(Left to right and Right to left)

- De Morgan's laws

$$\overline{A \cap B} = \bar{A} \cup \bar{B} \qquad \overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\bar{A} \cup \bar{B} = \overline{A \cap B} \qquad \bar{A} \cap \bar{B} = \overline{A \cup B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A = A \cap (A \cup B)$$

$$A = A \cup (A \cap B) \qquad A = A \cap (A \cup B)$$

- Complement laws

$$A \cup \bar{A} = U \qquad A \cap \bar{A} = \emptyset$$

$$U = A \cup \bar{A} \qquad \emptyset = A \cap \bar{A}$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and

2) $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

(Want to show: if x is in $\overline{A \cap B}$, then it must also be in $\bar{A} \cup \bar{B}$)

Assume $x \in \overline{A \cap B}$

$x \notin A \cap B$ By the definition of complement

$\neg((x \in A) \land (x \in B))$ by definition of intersection

$\neg(x \in A) \lor \neg(x \in B)$ by De Morgan's law

$(x \notin A) \lor (x \notin B)$ by definition of negation

$x \in \bar{A} \lor x \in \bar{B}$ by definition of the complement of a set

$x \in \bar{A} \cup \bar{B}$ by definition of union

We have shown that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

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Proof of Second De Morgan Law

These steps show that: $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

(Want to show: if x is in $\bar{A} \cup \bar{B}$, then it must also be in $\overline{A \cap B}$)

Assume $x \in \bar{A} \cup \bar{B}$

$(x \in \bar{A}) \vee (x \in \bar{B})$ *by* definition of union

$(x \notin A) \vee (x \notin B)$ by definition of complement

$\neg(x \in A) \vee \neg(x \in B)$ by definition of negation

$\neg((x \in A) \wedge (x \in B))$ De Morgan's law

$\neg(x \in A \cap B)$ by definition of intersection

$x \in \overline{A \cap B}$ by definition of complement

We have shown that : $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$ by definition of complement

$= \{x \mid \neg(x \in (A \cap B))\}$ by definition of "does not belong" symbol ($\notin$)

$= \{x \mid \neg(x \in A \wedge x \in B)\}$ by definition of intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ by De Morgan law for logical equival.

$= \{x \mid x \notin A \vee x \notin B\}$ by definition of "does not belong" symbol ($\notin$)

$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$ by definition of complement

$= \{x \mid x \in (\overline{A} \cup \overline{B})\}$ by definition of union

$= \overline{A} \cup \overline{B}$ by meaning of set builder notation

Membership Table

Use a membership table to show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
0	**0**	**0**	0	0	0	0	0
0	**0**	**1**	1	0	0	0	0
0	**1**	**0**	1	0	0	0	0
0	**1**	**1**	1	0	0	0	0
1	**0**	**0**	0	0	0	0	0
1	**0**	**1**	1	1	0	1	1
1	**1**	**0**	1	1	1	0	1
1	**1**	**1**	1	1	1	1	1

Example

Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

Solution:

$$\overline{A \cup (B \cap C)}$$

$$= \bar{A} \cap \overline{(B \cap C)} \quad \text{De Morgan law}$$

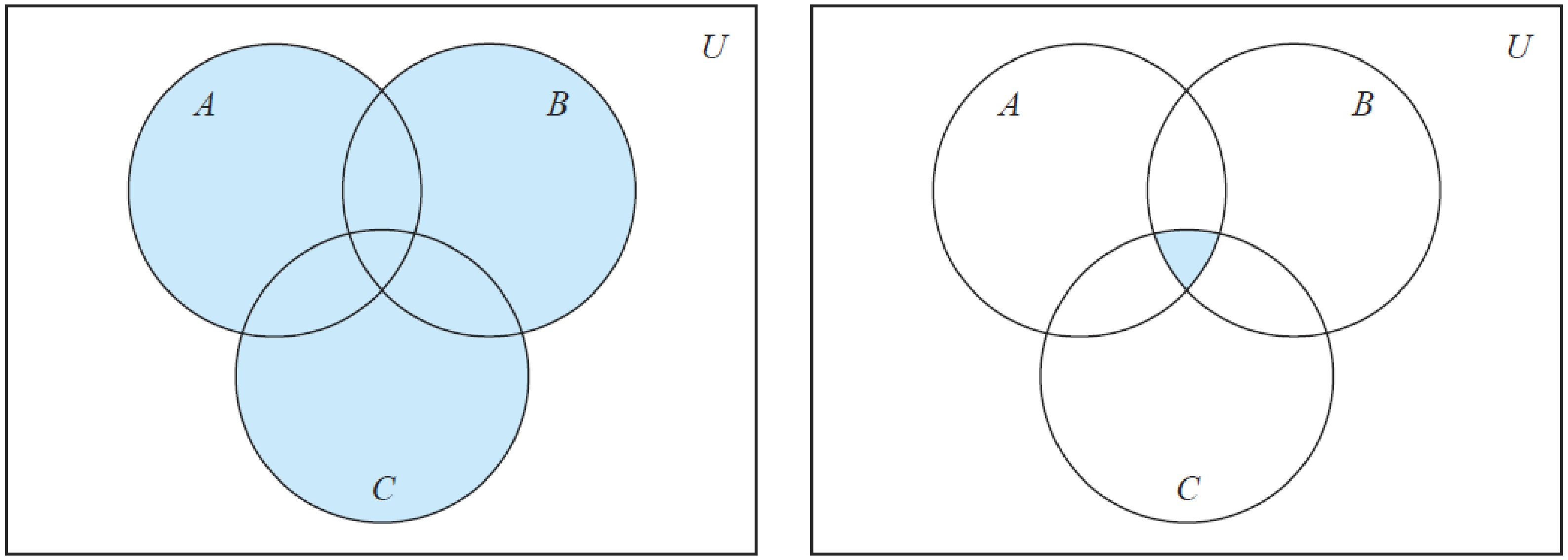
$$= \bar{A} \cap (\bar{B} \cup \bar{C}) \quad \text{De Morgan law}$$

$$= (\bar{B} \cup \bar{C}) \cap \bar{A} \quad \text{Commutative law}$$

$$= (\bar{C} \cup \bar{B}) \cap \bar{A} \quad \text{Commutative law}$$

This completes the proof

Generalized union and intersection



(a) $A \cup B \cup C$ is shaded.   (b) $A \cap B \cap C$ is shaded.

$A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, $C = \{0, 3, 6, 9\}$

$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$

$A \cap B \cap C = \{0\}$

Generalized Unions and Intersections

- Let A_1, A_2 , ..., A_n be an indexed collection of sets

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Computer representation of sets

- $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $A = \{1, 3, 5, 7, 9\}$ (odd integer ≤ 10),
  $B = \{1, 2, 3, 4, 5\}$ (integer ≤ 5)
- Represent A as 1010101010, and B as 1111100000
- Complement of A: 0101010101
- $A \cap B$: 1010101010 $\wedge$ 1111100000 = 1010100000

  which corresponds to $\{1, 3, 5\}$

$$\begin{array}{cccccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10\} \end{array}$$