7.2 Probability Theory

Credit

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Definition

- Suppose S is a set with n elements. The uniform distribution assigns the probability $1/n$ to each element of S.

- The experiment of selecting an element from a sample space S with a uniform distribution is called selecting an element of S at random.

Alternative definition

The probability of the event E is the sum of the probabilities of the outcomes in E. Thus

$$p(E) = \sum_{s \in E} p(s)$$

Note that when E is an infinite set,

$$\sum_{s \in E} p(s)$$

is a convergent infinite series

Probability

If A is a subset of S, let $\bar{A}$ be the complement of A with respect to S.

Then

$$p(\bar{A}) = 1 - p(A)$$

If A and B are subsets of S, then

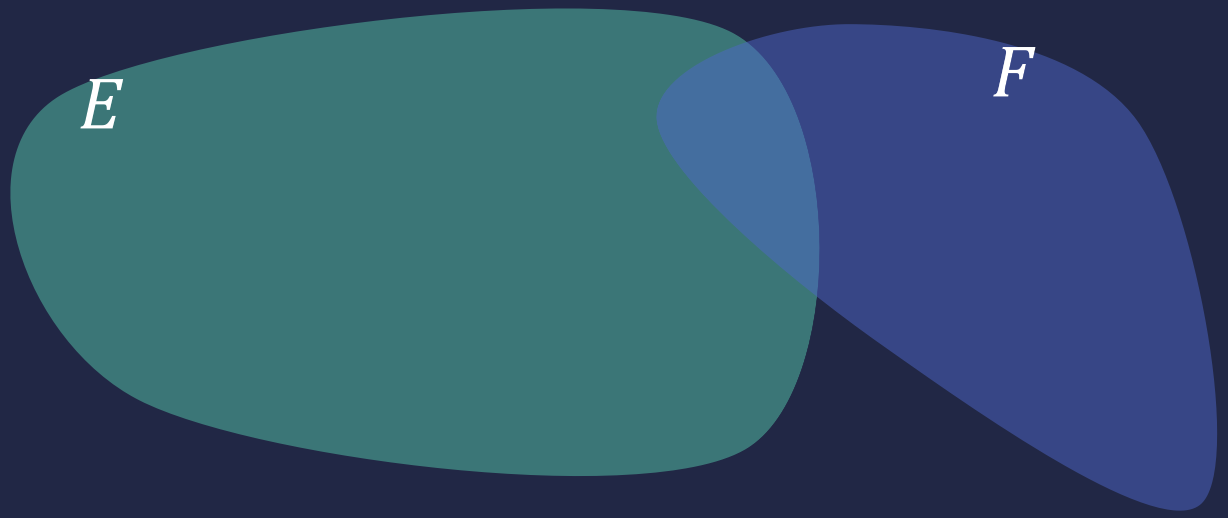
$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

Inclusion-Exclusion

Conditional Probability

Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $p(E|F)$ is defined to be:

$$p(E|F) = p(E \cap F) \, / \, p(F).$$



Example: Conditional Probability

A bit string of length 4 is generated at random so that each of the 16 bit possible strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

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Need to calculate

$$p(E|F)$$

Where F is the event that "first bit is 0", and E the event that "string contains at least two consecutive 0s".

What is "the experiment"?

The random generation of a 4 bit string.

What is the "sample space"?

The set of all possible outcomes, i.e., 16 possible strings. (equally likely)

Need to calculate $p(E|F)$ Where F is the event that "first bit is 0", and E the event that "string contains at least two consecutive 0s".

$$p(E|F) = p(E \cap F) \;/\; p(F)$$

sample space
$2^4 = 16$

$$p(E) = \frac{8}{16} = \frac{1}{2}$$

E Outcomes

0000 0001 0010 0011 0100 1000 1001 1100

$$p(F) = \frac{8}{16} = \frac{1}{2}$$

F outcomes

0000 0001 0010 0011 0100 0101 0110 0111

$$p(E \cap F) = \frac{5}{16}$$

$p(E \cap F)$ outcomes

0000 0001 0010 0011 0100 (note: 1st bit fixed to 0)

$p(E|F) = p(E \cap F) \;/\; p(F) = (5/16) \;/\; (1/2)$

$p(E|F) = 5/8$

Two Questions

A bit string of length 4 is generated at random so that each of the 16 bit strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

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F is the event that "first bit is 0",

E is the event that "string contains at least two consecutive 0s".

So, to calculate:

$$p(F|E) = p(F \cap E) \,/\, p(E)$$
$$= (p(E|F)\, p(F)) \,/\, p(E)$$

From $p(E|F) = p(E \cap F) \,/\, p(F)$

$p(E|F)\, p(F) = p(E \cap F)$

$p(\mathbf{E \cap F}) = p(E|F)\, p(F) = p(\mathbf{F \cap E})$

From previous example we had:

$p(E \cap F) = 5/16$ $\quad$ $p(E|F) = 5/8$ $\quad$ $p(F) = ½$ $\quad$ $p(E) = ½$

So, $p(F|E) = p(F \cap E) \,/\, p(E) = (5/16) \,/\, (1/2) = 5/8$

Also $p(F|E) = (p(E|F)\, p(F)) \,/\, p(E) = ((5/8) \cdot (1/2)) \,/\, (1/2) = 5/8$

Sample space	F	E	$(E \cap F)$	$E \mid F$	$F \mid E$
0000	0000	0000	0000	0000	0000
0001	0001	0001	0001	0001	0001
0010	0010	0010	0010	0010	0010
0011	0011	0011	0011	0011	0011
0100	0100	0100	0100	0100	0100
0101	0101			0101	
0110	0110			0110	
0111	0111			0111	
1000		1000			1000
1001		1001			1001
1010					
1011					
1100		1100			1100
1101					
1110					
1111					

$p(F) = ½$

$p(E) = ½$

$p(E \cap F) = 5/16$

$p(E|F) = 5/8$

$p(F|E) = 5/8$

F is the event that "first bit is 0",
E the event that "string contains at least two consecutive 0s".

Independence

The events E and F are independent if and only if

$$p(E \cap F) = p(E)\, p(F)$$

Note that in general: $p(E \cap F) = p(E)\, p(F|E)$

So, independent iff $p(F|E) = p(F)$

$(p(F|E) = p(E \cap F) / p(E) = (p(E)\, p(F)) / p(E) = p(F))$

Example: p("Tails" | "It's raining outside") = p("Tails").

Independence

The events E and F are independent if and only if

$$p(E \cap F) = p(E)\ \ p(F)$$

Let E be the event that a family of n children has children of both sexes.

Let F be the event that a family of n children has at most one boy.

Are E and F independent if $n = 2$?

No

Why?

$S = \{(\text{b, b}), (\text{b, g}), (\text{g ,b}), (\text{g, g})\}$, a family of 2 children
$E = \{(\text{b, g}), (\text{g, b})\}$, and $F = \{(\text{b, g}), (\text{g, b}), (\text{g, g})\}$

b for boy
g for girl

So $p(E \cap F) = 2/4 = 1/2$ and $p(E) \bullet p(F) = (1/2)\ (3/4) = 3/8$

Independence

The events E and F are independent if and only if
$p(E \cap F) = p(E)\, p(F)$

Let E be the event that a family of n children has children of both sexes.

Let F be the event that a family of n children has at most one boy.

Are E and F independent if $n = 3$?

Yes

Who will show us on the board?

$S = \{(b,b,b), (b,b,g), (b,g,b), (g,b,b), (g,g,b), (g,b,g), (b,g,g), (g,g,g)\}$
$E = \{(b, b, g), (b, g, b), (g, b, b), (g, g, b), (g, b, g), (b, g, g)\}$
$F = \{(g, g, b), (g, b, g), (b, g, g), (g, g, g)\}$
So $p(E \cap F) = 3/8$ and $p(E) \bullet p(F) = (6/8)\ (4/8) = 3/8$

Independence

The events E and F are independent if and only if $p(E \cap F) = p(E)\, p(F)$

Let E be the event that a family of n children has children of both sexes.

Let F be the event that a family of n children has at most one boy.

Are E and F independent if

$n = 4$? No

$n = 5$? No

So, dependence / independence depends on detailed structure of the underlying probability space and events in question!! (often the only way is to "calculate" the probabilities to determine dependence / independence).

Independence

Let E be the event that a family of 4 children has children of both sexes.

Let F be the event that a family of 4 children has at most one boy.

S = {(b,b,b,b), (b,b,b,g), (b,b,g,b), (b,g,b,b), (g,b,b,b), (b,b,g,g), (g,b,b,g), (g,g,b,b), (g,b,g,b), (b,g,g,b), (b,g,b,g), (b,g,g,g), (g,b,g,g), (g,g,b,g), (g,g,g,b), (g,g,g,g)} 16

E = {(b,b,b,g), (b,b,g,b), (b,g,b,b), (g,b,b,b), (b,b,g,g), (g,b,b,g), (g,g,b,b), (g,b,g,b), (b,g,g,b), (b,g,b,g), (b,g,g,g), (g,b,g,g), (g,g,b,g), (g,g,g,b)}

14 $p(E) = 14/16$

F = {(g, g, g, b), (g, g, b, g), (g, b, g, g), (b, g, g, g), (g, g, g, g)} 5 $p(F) = 5/16$

So $p(E \cap F) = 4/16$ and $p(E) \bullet p(F) = (14/16)\ (5/16) = 35/128$

$35/128 \neq 4/16$ so the two events are not independent

Bernoulli Trials

- A ***Bernoulli trial*** is an experiment, like flipping a coin, where there are ***two possible outcomes***. The probabilities of the two outcomes could be different.
- Example of ***two possible outcomes:***
  - ***Head*** **or** ***Tail***
  - ***Success*** **or** ***Fail***
  - ***1*** **or** ***0***

Bernoulli Trials

A coin is tossed 8 times.

(One face of the coin is **HEAD** and the other face is **TAIL**)

What is the probability of exactly 3 **HEADs** in the 8 tosses?

THHTTHTT is an example of tossing sequence with exactly 3 HEADs (**T** for TAIL and **H** for HEAD)

How many ways of choosing 3 positions for the **HEADs**?

$C(8, 3)$

What is the probability of a particular sequence?

(Assuming unbiased coin (fair))

0.5^8

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

$$C(n, k)\, p^k\, (1 - p)^{n - k}$$

Bernoulli Trials and Binomial Distribution

Bernoulli Formula: Consider an experiment which repeats a Bernoulli trial n times. Suppose each Bernoulli trial has possible outcomes A, B with respective probabilities p and $1 - p$. The probability that A occurs exactly k times in n trials is

$$C(n, k)\, p^k\, (1 - p)^{n - k}$$

Binomial Distribution: denoted by $b(k; n, p)$ – this function gives the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$

$$b(k; n, p) = C(n, k)\, p^k\, (1 - p)^{n - k}$$

$$C(n, r) = \frac{n!}{(n-r)!\,r!}$$

Bernoulli Trials

Consider flipping a fair coin *n* times.

A = coin comes up “heads”

B = coin comes up “tails”

$p = 1 - p = \mathbf{1/2}$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Q: What is the probability of getting exactly 10 heads if you flip a coin 20 times?

Recall: $p(A$ occurs k times out of $n)$

$$= \boldsymbol{C(n,k)\, p^k (1-p)^{n-k}}$$

$C(20,10)\,(1/2)^{10}\ (1/2)^{10}$

$= \quad C(20,10) \,/\, 2^{20}$

$= \quad 184756 \,/\, 1048576 = 0.1762\ldots$

Suppose a 0 bit is generated with probability 0.9 and a 1 bit is generated with probability 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits out of ten bits are generated?

$C(n, k)\, p^k\, (1 - p)^{n - k}$

b(8; 10, 0.9) = C(10,8) $(0.9)^8\,(0.1)^2$ = 0.1937102445

Bernoulli Trials

Self Study

A game of Jewel Quest is played 5 times. You clear the board 70% of the time (Win). What is the probability that you win a majority of the 5 games?

W: Win (clear the board)

L: Loose

Sanity check: What is the probability the result is WWLLW (3 Wins & 2 Looses – any order)?

$(0.7)^3\ (0.3)^2$

Assumes independent trials

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is $C(n, k)\ p^k\ (1 - p)^{n - k}$

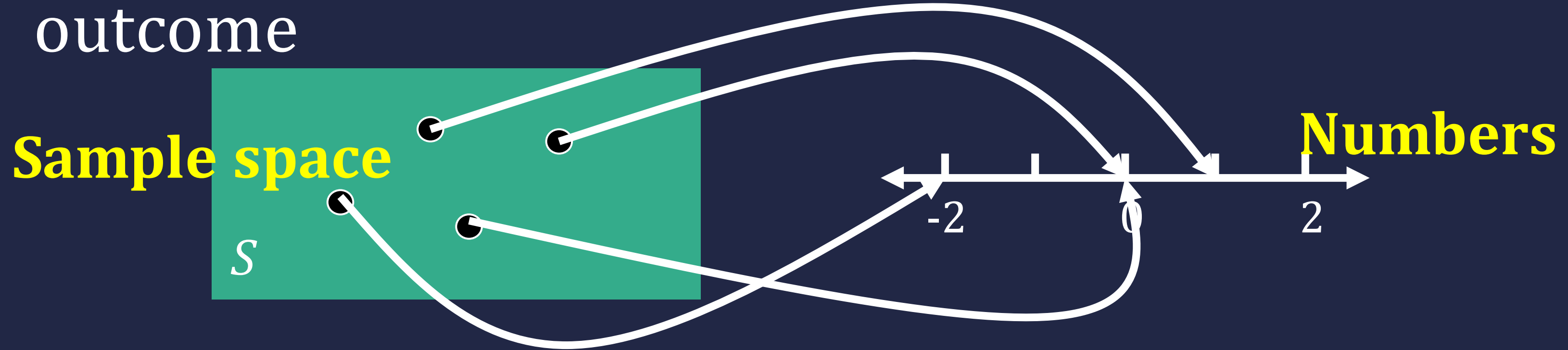
Win a majority of the 5 games: Win 3 games, 4 games, or 5 games

Win a majority of the 5 games =

$C(5,3)0.7^3 0.3^2 + C(5,4)0.7^4 0.3^1 + C(5,5)0.7^5 0.3^0$

Random Variables

For a given sample space S, a random variable (r.v.) is any real valued function on S, i.e., a random variable is a function that assigns a real number to each possible outcome



Suppose our experiment is a roll of 2 dice. S is set of pairs.

Example random variables:

X = sum of two dice. $X((2, 3)) = 5$

Y = difference between two dice. $Y((2, 3)) = 1$

Z = max of two dice. $Z((2, 3)) = 3$

r.v. Is a function not a variable and it is not random

Example: Random variable

Suppose a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome.

$X(\text{HHH}) = 3$

$X(\text{HHT}) = X(\text{HTH}) = X(\text{THH}) = 2$

$X(\text{TTH}) = X(\text{THT}) = X(\text{HTT}) = 1$

$X(\text{TTT}) = 0$

(**T** for **T**AIL and **H** for **H**EAD)

Note: we generally drop the argument! We just say the "random variable X"

And write e.g. $p(X = 2)$ for "the probability that the random variable $X(t)$ takes on the value 2".

Or $p(X = x)$ for "the probability that the random variable $X(t)$ takes on the value x."

Distribution of Random Variable

Definition:

The distribution of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r

A distribution is usually described specifying the probability $p(X = r)$ for each $r \in X(S)$

A probability distribution on a r.v. X is just an allocation of the total probability mass, 1, over the possible values of X

Example: Random variable

Suppose a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome.

X(HHH) = 3

X(HHT) = X(HTH) = X(THH) = 2

X(TTH) = X(THT) = X(HTT)= 1

X(TTT) = 0

(**T** for **T**AIL and **H** for **H**EAD)

when a fair coin is flipped three times each of the eight possible outcomes has probability 1/8.

- The distribution of the random variable $X(t)$ is determined by the probabilities $P(X = 3) = 1/8$, $P(X = 2) = 3/8$, $P(X = 1) = 3/8$, and $P(X = 0) = 1/8$.

Consequently, the distribution of $X(t)$ is the set of pairs

{(3, 1/8), (2, 3/8), (1, 3/8), and (0, 1/8)}.

Birthdays

How many people have to be in a room to assure that th

probability that at least two of them have the same birthday is greater than 1/2?

a) 23
b) 183
c) 365
d) 730
e) Non of the above

Let p_n be the probability that no people share a birthday among n people in a room.

a: 23

Then $1 - p_n$ is the probability that 2 or more share a birthday.

We want the smallest n so that $1 - p_n > 1/2$.

Birthdays

Assumption:

Birthdays of the people are independent

Each birthday is equally likely and that there are 366 days/year

Let p_n be the probability that *no-one* shares a birthday among n people in a room. *[Each person has a different birthday than the others]* What is p_n?

Assume that people come in certain order;
the probability that the second person has a birthday different than the first is (366 - (2 - 1))/366 = 365/366;
the probability that the **third** person has a different birthday form the two previous ones is (366 - (**3** - 1))/366 = 364/366.
For the ***jth*** person we have (366 - (***j*** - 1))/366

So, $p_n = \frac{365}{366} \cdot \frac{364}{366} \cdot \frac{363}{366} \cdot \ldots \cdot \frac{367-n}{366}$

$$1 - p_n = 1 - \frac{365}{366} \cdot \frac{364}{366} \cdot \frac{363}{366} \cdot \ldots \cdot \frac{367-n}{366}$$

After several tries, when $n = 22$ $1 - p_n = 0.475$.

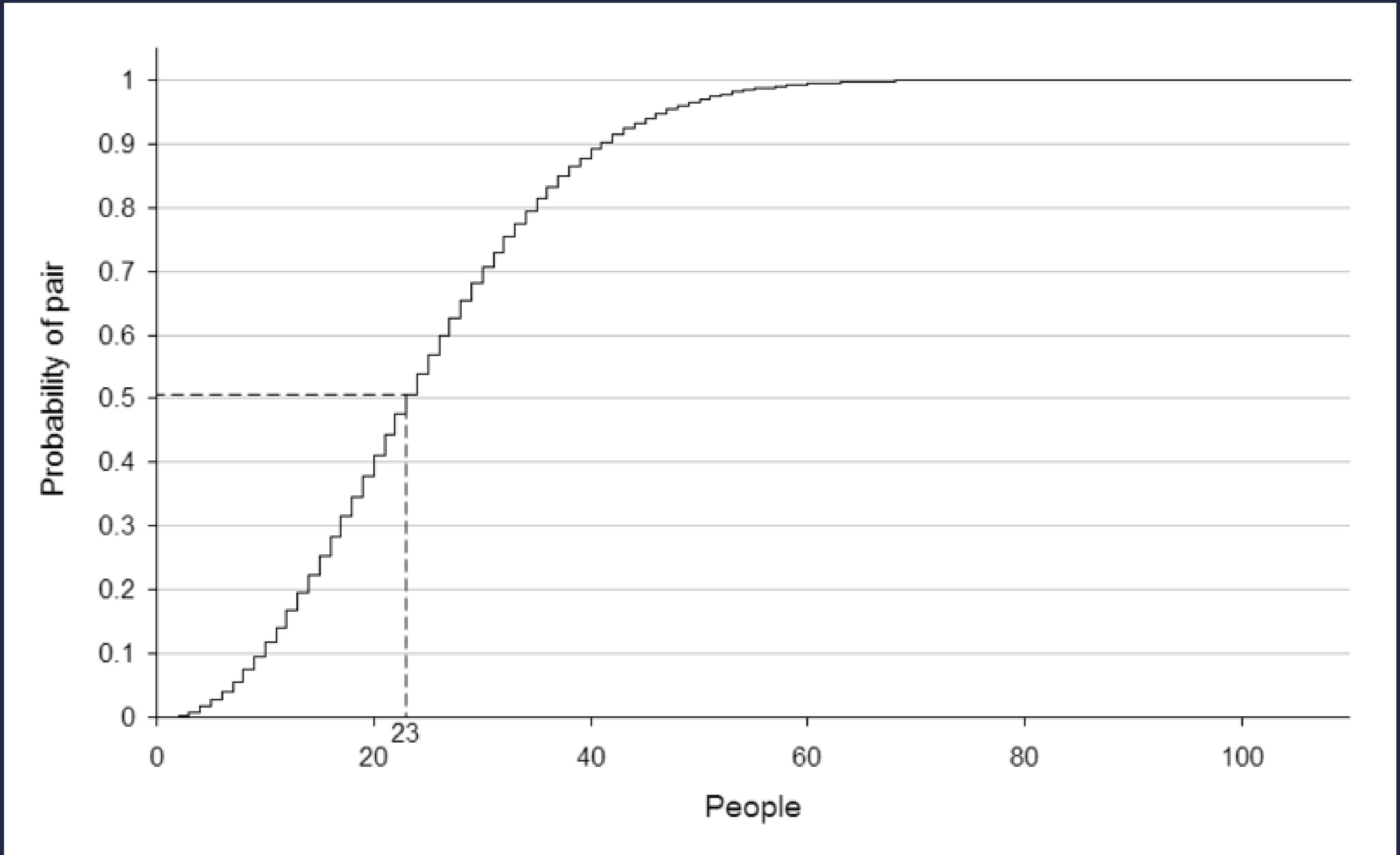
$n = 23$ and $1 - p_n = 0.506$

Relevant to “hashing”. What is “Hashing?”

Verify as an exercise

- About a 3% chance that in a group of 5 people at least two people share the same birthday.
- About a 51% chance that in a group of 23 people at least two people share the same birthday.
- About a 97% chance that in a group of 50 people at least two people share the same birthday.

The Birthday Problem



The Birthday Problem (assuming **365** birthdays)

n	p(n)
1	0.0%
5	2.7%
10	11.7%
20	41.1%
23	50.7%
30	70.6%
40	89.1%
50	97.0%
60	99.4%

n	p(n)
70	99.9%
100	99.99997%
200	99.99999999999999999999999999998%
300	$(100 - (6\times10^{-80}))\%$
350	$(100 - (3\times10^{-129}))\%$
365	$(100 - (1.45\times10^{-155}))\%$
366	100%
367	100%