6.4 Binomial Coefficients and Identities

Credit

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Powers of Binomial Expressions

Definition:

a **binomial** expression is the sum of two terms, such as $x + y$.

More generally, these terms can be products of constants and variables.

Powers of Binomial Expressions

- We first look at the process of expanding
- $(x + y)^3 = (x + y)\,(x + y)\,(x + y)$
- Terms x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?

Expanding $(x+y)^3 = (x+y)(x+y)(x+y)$

- What are the coefficients of x^3, x^2y, xy^2, y^3
  - To obtain $\boldsymbol{x^3}$, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is **1**.

$$\binom{3}{2} = \frac{3!}{(3-2)!\,2!}$$

  - To obtain $\boldsymbol{x^2y}$, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is **3**.

Expanding $(x+y)^3 = (x+y)(x+y)(x+y)$ [Continue]

$$\binom{3}{1} = \frac{3!}{(3-1)!\,1!}$$

- To obtain $\boldsymbol{xy^2}$, an x must be chosen from one of the sums and a y from the other two. There are $\binom{3}{2}$ ways to do this and so the coefficient of xy^2 is **3**.
- To obtain $\boldsymbol{y^3}$, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is **1**.

- This is a counting argument to show that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

Expanding $(x+y)^3 = (x+y)\,(x+y)\,(x+y)$ [Continue]

$$\binom{3}{1} = \frac{3!}{(3-1)!\,1!}$$

$$(x+y)^3 = (xx + xy + yx + yy)(x+y)$$

$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

- This is a counting argument to show that

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^3 = C(3\,,3)\,x^3 + C(3\,,2)\,x^2y + C(3\,,1)\,xy^2 + C(3\,,0)\,y^3$$

$$(x+y)^3 = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

Notice that $\binom{3}{3} = \binom{3}{0}$ and $\binom{3}{2} = \binom{3}{1}$

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

Binomial Theorem

Binomial Theorem: let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Using the Binomial Theorem

Example: what is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Solution: we view the expression as $((2x) + (-3y))^{25}$

By the binomial theorem $((2x) + (-3y))^{25} =$

$$\sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

$$= \sum_{j=0}^{25} \binom{25}{j} (2)^{25-j} (x)^{25-j} (-3)^j (y)^j$$

$$= \sum_{j=0}^{25} \binom{25}{j} (2)^{25-j} (-3)^j (x)^{25-j} (y)^j$$

The coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13! \cdot 12!} \cdot 2^{12} \cdot 3^{13}$$

Binomial Coefficients: Example

What is the coefficient of the term x^8y^{12} in the expansion of $(3x + 4y)^{20}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

By the binomial theorem, we have

$$(3x+4y)^{20} = \sum_{j=0}^{20} \binom{20}{j} (3x)^{20-j} (4y)^j$$

When $j = 12$, we have $\binom{20}{12}(3x)^8(4y)^{12}$

The coefficient is

$$\binom{20}{12} 3^8 \cdot 4^{12} = \frac{20!}{12! \cdot 8!} \cdot 3^8 \cdot 4^{12} = 13866187326750720$$

Binomial Coefficients

- Many useful identities and facts come from the Binomial Theorem

2^n

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

based on $(1 + 1)^n = 2^n$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\,, n \geq 1$$

based on $((-1) + 1)^n = 0^n$

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

based on $(1 + 2)^n = 3^n$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

A Useful Identity

Corollary: with $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: with $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{(n-k)} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Binomial Coefficients

$0^n = (1 + (-1))^n =$

$x = 1, y = -1$

$$\sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (-1)^k =$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k, n \geq 1$$

Binomial Theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Binomial Coefficients

$x = 1, y = 2$

$3^n = (1 + 2)^n$

$$3^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (2)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Binomial Theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Binomial Coefficients: Pascal’s Identity & Triangle

- Pascal’s identity gives a useful identity for efficiently computing binomial coefficients
- **Theorem**: Pascal’s Identity

  Let $n, k \in \mathbb{Z}^+$ with $n \geq k$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal’s Identity forms the basis of a geometric object known as Pascal’s Triangle

Pascal's Triangle

Each n-th row binomial coefficients

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

n

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

k

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

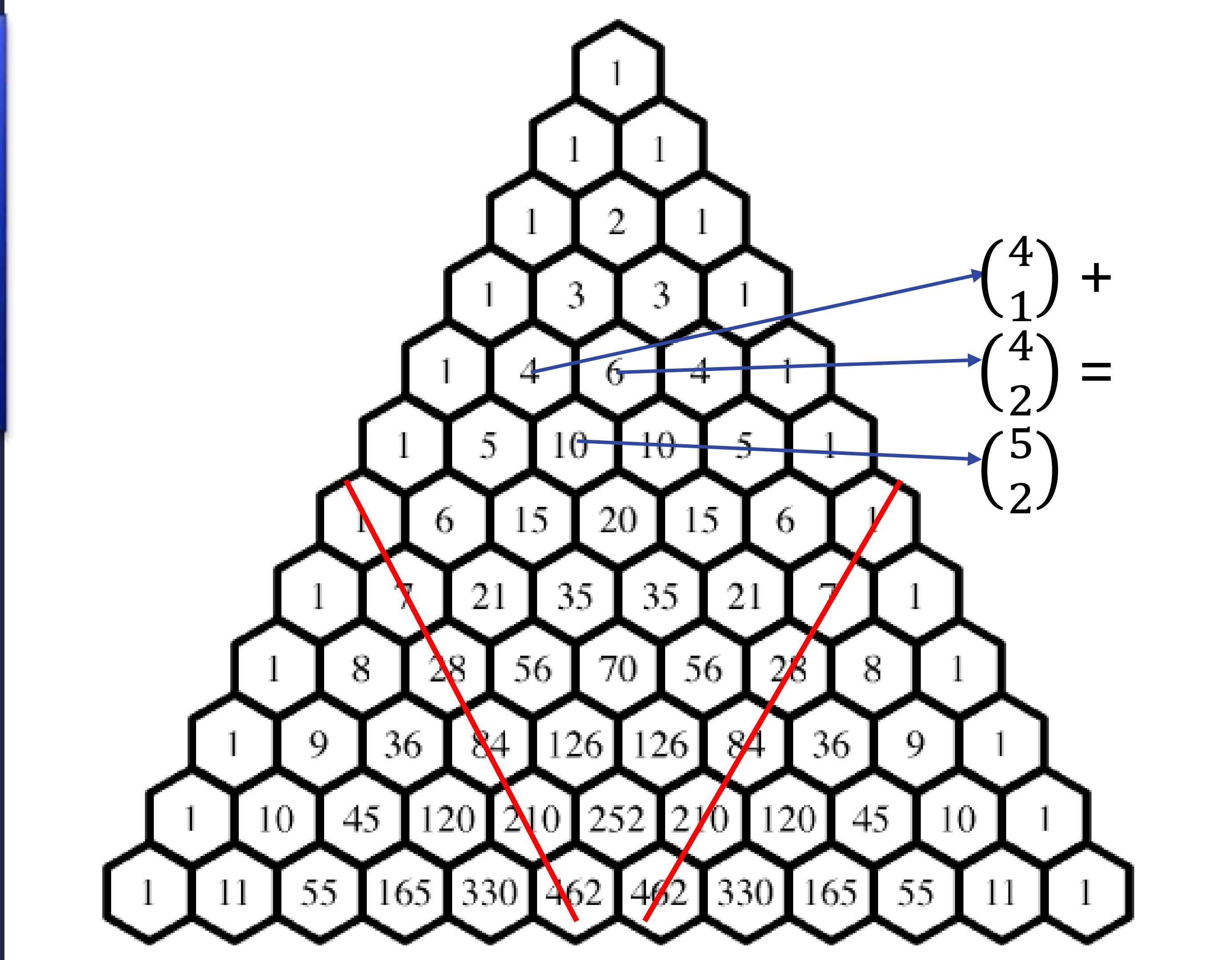
This leads to Pascal's Triangle

From this we see an easy way to generate all coefficients recursively.

Each n-th row binomial coefficients

$$\binom{n+1}{k}$$

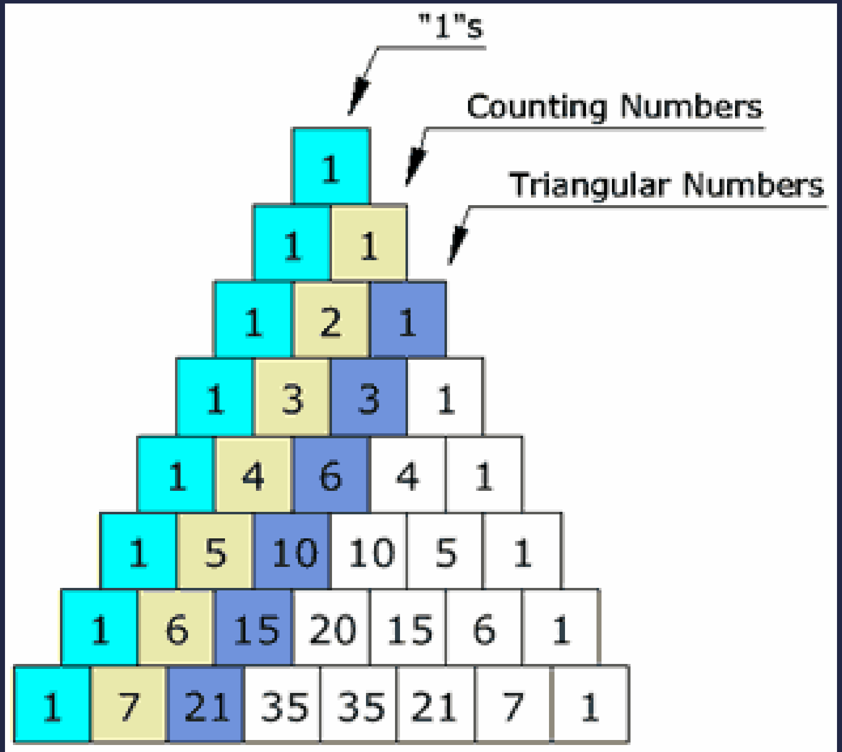
$$= \binom{n}{k-1} + \binom{n}{k}$$



$$\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$$

How to build Pascal triangle

To build the triangle:
Start with "1" at the top then continue placing numbers below it in a triangular pattern.
If it is left edge or right edge put 1.
If it is not an edge put the sum of the two numbers above it.



Triangular numbers: The series of numbers (1, 3, 6, 10, 15, etc.) obtained by continued summation of the natural numbers 1, 2, 3, 4, 5, etc.

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Creating Pascal's Triangle

$(x + y)^0 = 1$

$(x + y)^1 = 1x + 1y$

$(x + y)^2 = 1x^2 + 2xy + 1y^2$

$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Answer the following questions before going to the next slide:

1) **Where do the numbers come from to form the triangle?**
2) **How do you continue the triangle for three more rows?**

1) Where do the numbers come from to form the triangle?

The entries in the triangle are the coefficients of the terms of the binomial expression: $(x+y)^n$

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

2) How do you continue the triangle for three more rows?

In the triangle the first and last terms are always 1. The terms in between are the sum of two adjacent terms. (watch above).

The next three rows would be:

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Patterns in Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Look at the numbers in the ellipse.

Add the numbers two at a time.

Example: 1 + 3 = 4

3 + 6 = 9

6 +10 =16

The answers are all perfect squares.

What kind of answers do you get?

Try to find some other number patterns.

Patterns in Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Look at the part of a diagonal in the rectangle starting from the right edge.

Add the four numbers together.

$$1 + 3 + 6 + 10 = 20$$

Where do you find the answer in Pascal's Triangle?

Some people call this the Hockey Stick Pattern.

Try this for other parts of diagonals?

Combinations Using Formulas

- Suppose you have three socks and want to figure out how many different ways you can choose two of them to wear. You don't care which feet you put them on, it only matters which two socks you pick.
- This problem amounts to the question "how may different ways can you choose two objects from a set of three objects?"

Suppose you have S_1, S_2, S_3 (sock 1, sock 2, sock 3)

You could choose: S_1,S_2 S_1,S_3 S_2,S_3 (3 ways to choose 2 socks)

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$ Sometimes written as $C(n,r)$ or ${}_nC_r$

$\binom{3}{2} = C(3,2) = {}_3C_2$ = 3! / (2! · 1!) = (3 · 2 · 1) / (2 · 1 · 1) = 6 / 2 = 3

Combinations Using Formulas (continued)

Suppose you wanted to choose one sock.

S_1 or S_2 or S_3 (3 ways)

$C(3, 1) = \binom{3}{1} = 3! \,/\, 1!2! = 3$ ways

Suppose you wanted to choose all three socks

$\{S_1, S_2, S_3\}$ (1 way)

$C(3, 3) = \binom{3}{3} = 3! \,/\, 3! \cdot 0! = 1$ way

Combinations and Pascal's Triangle

Now use Pascal's Triangle to determine how many ways you can choose 1 sock, 2 socks, or 3 socks.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

1 way for choosing three socks

3 ways for choosing one sock

3 ways for choosing two socks

1) Ignore the 1's running down the left-hand side of Pascal's Triangle.

2) Choose the row where 3 is in the second position because you have 3 socks.

Compare using the previous formula method. $C(n,r) = \binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

A basketball coach is criticized in the newspaper for not trying out every combination of players. If the team roster has 10 players, how many five-player combinations are possible?

We can use the combination formula to find how many combinations of five players are possible.

$$C(10,5) = \binom{10}{5} = \frac{10!}{(10-5)!5!} = \frac{10!}{5!\,5!} = 252$$

ways to choose five players

Or...

Use Pascal's Triangle to find the number of ways.

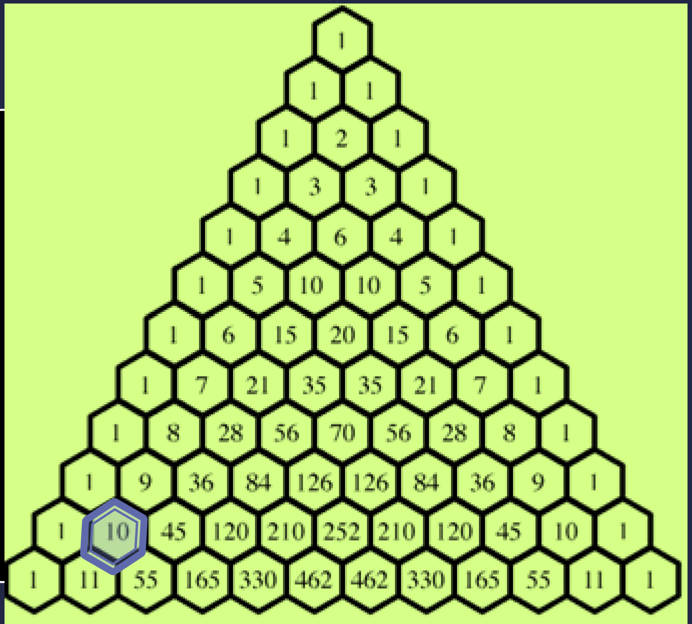
You choose the row with 10 after the 1 because there are 10 players from which to choose.

The row where 10 is the first number after the 1 is:

1 10 45 120 210 252 210 120 45 10 1

This row tells you there are:

10 ways to choose one player out of 10
45 ways to choose two players out of 10
120 ways to choose three players out of 10
210 ways to choose four players out of 10
252 ways to choose five players out of 10



$$C(10,5) = \binom{10}{5} = \frac{10!}{5!\,5!} = 252$$

Pascal's Triangle and the formula result in the same answer.

There are 8 people in an office. 3 are to be chosen for the grievance committee. In how many ways can 3 people be chosen out of the eight in the office?

You can use the combination formula to find how many combinations of three people are possible.

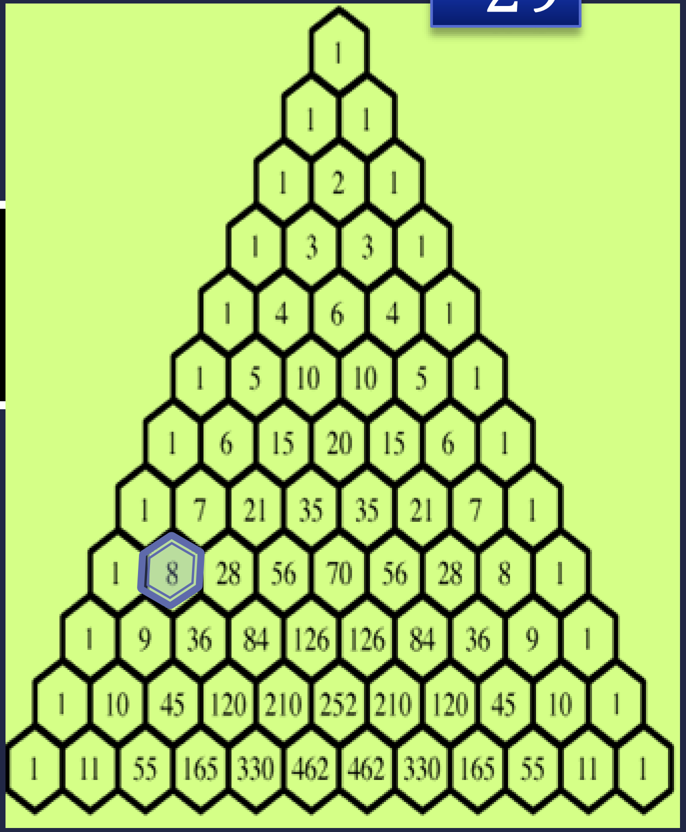
$C(8,3) = \binom{8}{3} = \frac{8!}{5!3!} =$ 56 ways to choose five players

$_8C_3$

OR …

Use Pascal's Triangle to find the number of ways.

You choose the first row that has an 8 after the one since there are 8 people to choose from.



The row where 8 (you have 8 people) is the first number after the 1:

1 8 28 56 70 56 28 8 1

This row of numbers tells you:

8 ways to choose one person out of 8

28 ways to choose two people out of 8

56 ways to choose three people out of 8

$$C(8,3) = \binom{8}{3} = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = 56$$

ways to choose five players

Pascal's triangle gives the same answer as the formula.

The Binomial Theorem

How do we expand these?

1. $(x + 2)^2$
2. $(2x + 3)^2$
3. $(x - 3)^3$
4. $(a + b)^4$

The Binomial Theorem

1. $(x + 2)^2$ **2.** $(2x + 3)^2$
3. $(x - 3)^3$ 4. $(a + b)^4$

1. $(x + 2)^2 = x^2 + 2(2)x + 2^2 = x^2 + 4x + 4$

2. $(2x + 3)^2 = (2x)^2 + 2(3)(2x) + 3^2 = 4x^2 + 12x + 9$

3. $(x - 3)^3 = (x - 3)(x - 3)^2 = (x - 3)(x^2 - 2(3)x + 3^2)$
$= (x - 3)(x^2 - 6x + 9) = x(x^2 - 6x + 9) - 3(x^2 - 6x + 9)$
$= x^3 - 6x^2 + 9x - 3x^2 + 18x - 27 = x^3 - 9x^2 + 27x - 27$

4. $(a + b)^4 = (a + b)^2(a + b)^2 = (a^2 + 2ab + b^2)(a^2 + 2ab + b^2)$
$= a^2(a^2 + 2ab + b^2) + 2ab(a^2 + 2ab + b^2) + b^2(a^2 + 2ab + b^2)$
$= a^4 + 2a^3b + a^2b^2 + 2a^3b + 4a^2b^2 + 2ab^3 + a^2b^2 + 2ab^3 + b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Easier way:

$$(x+y)^n =$$

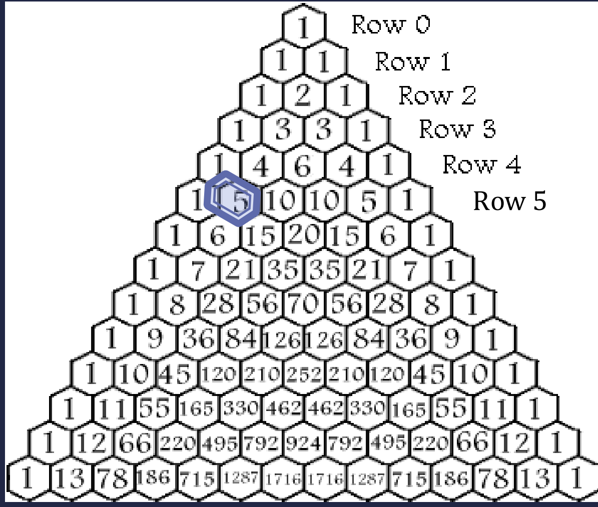
$$\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots$$

$$+ \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

The Binomial Theorem

Use Pascal's Triangle to expand $(a + b)^5$.

Use the row that has 5 as its second number.



The exponents for a begin with 5 and decrease.

$$1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

The exponents for b begin with 0 and increase.

In its simplest form, the expansion is
$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$(x+y)^n =$$

$$\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

The Binomial Theorem

Use Pascal's Triangle to expand $(x - 3)^4$.

First write the pattern for raising a binomial to the fourth power.

1 4 6 4 1 Coefficients from Pascal's Triangle.

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Since $(x - 3)^4 = (x + (-3))^4$, substitute x for a and -3 for b.

$$(x + (-3))^4 = x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4$$

$$= x^4 - 12x^3 + 54x^2 - 108x + 81$$

The expansion of $(x - 3)^4$ is $x^4 - 12x^3 + 54x^2 - 108x + 81$.

Use the Binomial Theorem to expand $(x - y)^9$

Write the pattern for raising a binomial to the ninth power.

$$(a+b)^9 = \binom{9}{0}a^9 + \binom{9}{1}a^8b + \binom{9}{2}a^7b^2 + \binom{9}{3}a^6b^3 + \binom{9}{4}a^5b^4 + \binom{9}{5}a^4b^5 + \binom{9}{6}a^3b^6 + \binom{9}{7}a^2b^7 + \binom{9}{8}ab^8 + \binom{9}{9}b^9$$

Substitute x for a and $-y$ for b. Evaluate each combination.

$$(x-y)^9 = \binom{9}{0}x^9 + \binom{9}{1}x^8(-y) + \binom{9}{2}x^7(-y)^2 + \binom{9}{3}x^6(-y)^3 + \binom{9}{4}x^5(-y)^4 + \binom{9}{5}x^4(-y)^5 + \binom{9}{6}x^3(-y)^6 + \binom{9}{7}x^2(-y)^7 + \binom{9}{8}x(-y)^8 + \binom{9}{9}(-y)^9$$

$$= x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$$

The expansion of $(x - y)^9$ is $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$.

Use the Binomial Theorem to expand $(x - y)^9$

Write the pattern for raising a binomial to the ninth power.

$(a + b)^9 = {}_9C_0a^9 + {}_9C_1a^8b + {}_9C_2a^7b^2 + {}_9C_3a^6b^3 + {}_9C_4a^5b^4 + {}_9C_5a^4b^5 + {}_9C_6a^3b^6 + {}_9C_7a^2b^7 + {}_9C_8ab^8 + {}_9C_9b^9$

Substitute x for a and $-y$ for b. Evaluate each combination.

$(x - y)^9 = {}_9C_0x^9 + {}_9C_1x^8(-y) + {}_9C_2x^7(-y)^2 + {}_9C_3x^6(-y)^3 + {}_9C_4x^5(-y)^4 + {}_9C_5x^4(-y)^5 + {}_9C_6x^3(-y)^6 + {}_9C_7x^2(-y)^7 + {}_9C_8x(-y)^8 + {}_9C_9(-y)^9$

$= x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$

The expansion of $(x - y)^9$ is $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$.

Using other notations for combinations: $\boldsymbol{{}_nC_r = C(n, r) = \binom{n}{r}}$