6.3 Permutations and Combinations

Credit

Richard Scherl, Ming-Hsuan Yang,

Mehrdad Nojoumian, Max Welling, R. A. Pilgrim,

Paul Kennedy, Mariam Beydoun, Arick Little,

Husni Al-Muhtaseb

Permutations

Definition: a **permutation** of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

Example: let $S = \{1, 2, 3\}$.

- The ordered arrangement 3, 1, 2 is a permutation of S.
- The ordered arrangement 3, 2 is a 2-permutation of S.

Permutations

- The number of r-permutations of a set with n elements is denoted by $P(n, r)$.
  - The 2-permutations of $S = \{1, 2, 3\}$ are

    1, 2 ; 1, 3 ; 2, 1 ; 2, 3 ; 3, 1 ; 3, 2

    Hence, $P(3, 2) = 6$

Permutations

Theorem: if n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary: if n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutations

Example: how many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: $P(100, 3) = 100 \times 99 \times 98 = 970{,}200$

Permutations

- Example: suppose that a salesman has to visit 8 different cities. He must begin his trip in a specified city, but he can visit the other seven cities in any order he wishes. How many possible orders can the salesman use when visiting these cities?
  - Solution: the first city is chosen, and the rest are ordered arbitrarily:
    - $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$
  - If he wants to find the tour with the shortest path that visits all the cities, he must consider 5040 paths!

Permutations

- Example: how many permutations of the letters $ABCDEFGH$ contain the string "ABC" ?
  - Solution: solve this by counting the permutations of six objects, ABC, D, E, F, G, H.
- That is,

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

Combinations

Definition: an ***r*-combination** of elements of a set S is an unordered selection of r elements from the set S. Therefore, an r-combination is simply a subset of the set with r elements.

- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

Combinations

Example: let S be the set $\{a, b, c, d\}$. Then the set $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

Example: $C\,(4\,,2) = 6$ or $\binom{4}{2} = 6$

because the 2-combinations of $\{a, b, c, d\}$ are the six subsets:

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}.$$

Combinations

Theorem: the number of r-combinations of a set with n elements, where $0 \le r \le n$, equals

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Proof: by the product rule

$P(n, r) = C(n, r) \times P(r, r)$. Therefore,

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{(n-r)!\,r!}$$

The r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations.

Combinations

Example: In how many ways can we pick 11 players from 21 candidates?

Solution: since the order in which the players are selected does not matter, the number is:

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$$C(21, 11) = \binom{21}{11} = \frac{21!}{10!\,11!}$$

Combinations

Example: How many bit-strings of size 10 contain four 1's.

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Solution: We need to place four 1's in 10 slots:

$$C\,(10, 4) = \binom{10}{4} = \frac{10!}{6!\,4!}$$

Combinations

Example: We need to form a committee of 7 people, 3 from math and 4 from computer science to develop a discrete math course. There are 9 math candidates and 11 computer science candidates. How many possibilities?

Solution: Two separate problems that need to be combined using the product rule.

$C\,(9, 3)$ possibilities for math AND $C\,(11, 4)$ possibilities for computer science:

Total = $C\,(9, 3)\; C\,(11, 4) = 27{,}720$.

Combinations

Corollary: let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$.

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof: from the previous Theorem, it follows that

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

and

$$C(n,n-r) = \frac{n!}{[n-(n-r)]!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

Hence, $C(n, r) = C(n, n - r)$.

Combinations

Example: how many ways are there to select 5 players from a 10-member tennis team to make a trip to a match.

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Solution: the number of combinations is

$$C(10,5) = \binom{10}{5} = \frac{10!}{5!\,5!} = 252$$

Combinations

Example: a group of 30 people have been trained to go on the first mission to Mars. How many ways are there to select a crew of 6 people to go on this mission?

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Solution: the number of possible crews is

$$C(30,6) = \binom{30}{6} = \frac{30!}{24!\,6!}$$

$$= \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593{,}775$$