5.1 Mathematical Induction

Credit

Ming-Hsuan Yang

Nasir Al-Darwish

Husni Al-Muhtaseb

Structure of a Proof by Induction

Induction can be used to prove that a given proposition, $P(n)$, holds for all integers $n \geq n_0$, where n_0 is some fixed integer.

The proof consists of two steps:

Base step:

Prove $P(n_0)$

Induction step:

Prove

$$\forall n \geq n_0\ P(n) \to P(n+1)$$

Structure of a Proof by Induction

The proof consists of two steps:

Base step: Prove $P(n_0)$

Induction step: Prove

$$\forall n \geq n_0$$

$$P(n) \rightarrow P(n+1)$$

In other words, for an arbitrary integer n (where $n \geq n_0$) we assume that $P(n)$ is true and show as a consequence that $P(n+1)$ is true.

The left side of the above implication($P(n)$) is called the *induction hypothesis* (IH) because it is what is assumed in the induction step.

Structure of a Proof by Induction

The induction step

Prove $\forall n \geq n_0\ P(n) \to P(n+1)$

is also equivalent to:

Prove $\forall n > n_0\ P(n-1) \to P(n)$

(Note the inequality and the $\pm\mathbf{1}$*)*

and

Prove $\forall k \geq n_0\ P(k) \to P(k+1)$

(Can use any letter instead of n)

Why Induction Proof is Valid

An induction proof establishes the following propositions:

$P(n_0)$, $P(n_0 + 1)$, $P(n_0 + 2)$, ... and so on.

First note that the induction step establishes:

$P(n_0) \rightarrow P(n_0 + 1)$,

$P(n_0 + 1) \rightarrow P(n_0 + 2)$,

...

and so on.

Why Induction Proof is Valid

$P(n_0)$ is established by the base step

$P(n_0 + 1)$ is established by (modus ponens)

$P(n_0) \to P(n_0 + 1)$, and $P(n_0)$

$P(n_0 + 2)$ is established by

$P(n_0 + 1) \to P(n_0 + 2)$ and $P(n_0 + 1)$

$$\begin{array}{l} P(n_0) \to P(n_0 + 1) \\ P(n_0) \\ \hline P(n_0 + 1) \end{array}$$

This shows that the proof method is sound. It is being gradually established for each successive value of n.

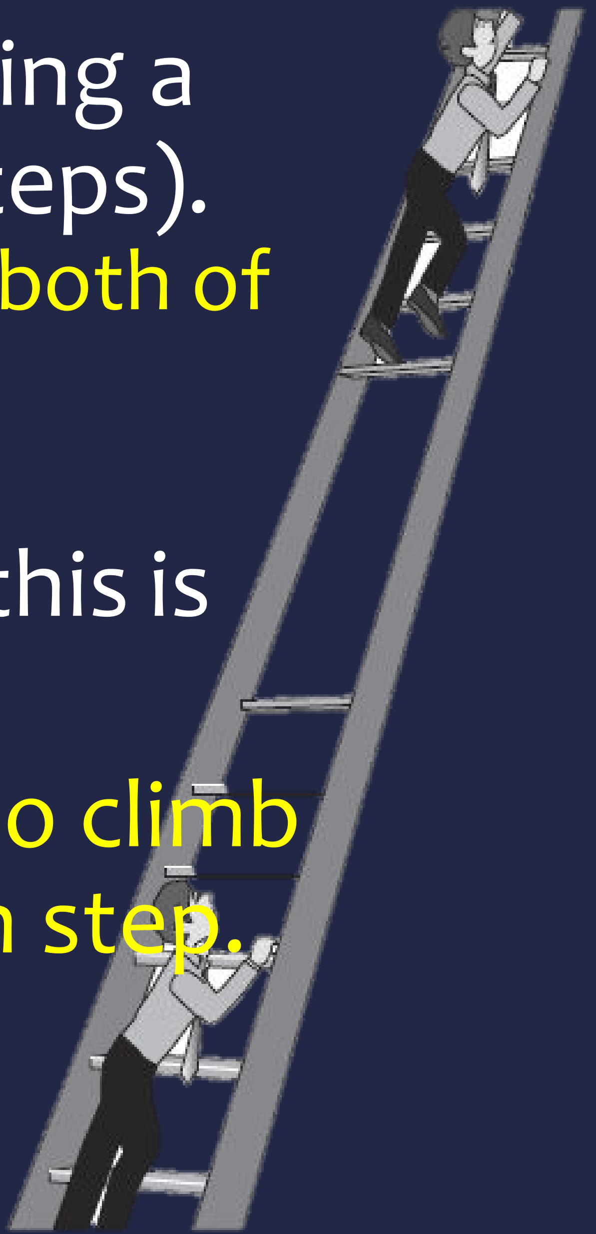
$$\begin{array}{l} P(n_0 + 1) \to P(n_0 + 2) \\ P(n_0 + 1) \\ \hline P(n_0 + 2) \end{array}$$

Why Induction Proof is Valid

A proof by induction is similar to climbing a ladder (having an infinite number of steps).

One is able to climb all the ladder steps if both of the following propositions are true:

1. He is able to climb to the first step; this is the base step.
2. From an arbitrary step *n*, he is able to climb one step higher; this is the induction step.



Example 1

Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

for all integers $n \geq 1$.

Ex 1: Prove that $1 + 2 + \ldots + n = n(n + 1) / 2$ for all integers $n \geq 1$.

Let $p(n)$ be $1 + 2 + \ldots + n = n(n + 1) / 2$ for all integers $n \geq 1$.

Base step: We show $P(n)$ for $n = 1$.
LHS = 1 and RHS = $1 (1 + 1) / 2 = 1$. Thus, $p(n)$ is true for $n = 1$.

Induction step: We must show that, for $n \geq 1$, $P(n) \rightarrow P(n + 1)$. Thus, we assume $P(n)$ is true (*IH*). That is: $1 + 2 + \ldots + n = n(n + 1) / 2$ **(1)**

We must show $P(n + 1)$. That is,

$1 + 2 + \ldots + n + (n + 1) = (n + 1) ((n + 1) + 1) / 2 = (n + 1) (n + 2) / 2$ **(2)**

LHS of Equation **(2)** = $1 + 2 + \ldots + n + (n + 1) = n(n + 1) / 2 + (n + 1)$,

where the sum of the first n terms on the LHS is replaced by the RHS of Equation **(1)**.

The latter expression = $(n + 1) (n / 2 + 1) = (n + 1) (n / 2 + 2 / 2)$
$= (n + 1) (n + 2) / 2$ = RHS of Equation **(2)**.

Induction Proof – Example 2

Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using induction.

1 = **1**, 1 + 3 = **4**, 1 + 3 + 5 = **9**, 1 + 3 + 5 + 7 = **16**,

1 + 3 + 5 + 7 + 9 = **25**, 1 + 3 + 5 + 7 + 9 + 11 = **36**

It is reasonable to conjecture that the sum of first n odd integers is n^2, that is, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

In the preceding proposition, n ranges over integers (and not odd integers). Thus, it is appropriate to use induction to prove it.

Induction Proof – Example 2 – Cont.

- Let $p(n)$ denote the proposition $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
- *Base step*: $p(1)$ (that is, $n = 1$): LHS = 1, RHS = 1^2 . LHS = RHS
- *Induction step*: Assume that $p(k)$ is true, i.e.,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

We must show $p(k + 1)$, that is,

$$1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2 \quad (1)$$

LHS of (1) = $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1)$

$= k^2 + 2k + 1 = (k + 1)^2$ = RHS of (1)

- We have completed both the base and induction ste
we have shown $p(1)$ is true and $p(k) \rightarrow p(k + 1)$. Cor
$p(n)$ is true for all positive integers *n.*

k	$p(k)$
1	1
2	3
3	5
…	
$k - 1$	$2(k-1)-1$ $= 2k - 3$
k	$2k - 1$
$k + 1$	$2(k+1)-1$ $= 2k + 1$

Induction Proof – Example 3

Prove by induction that for any real number $a \neq 1$ and all integers $n \geq 0$:

$$1 + a + a^2 + \ldots + a^n = (a^{n+1} - 1) / (a - 1)$$

Note that the terms on LHS form a *geometric progression*, where every term is obtained from the previous term by multiplying by some fixed factor a.

Let $p(n)$ be the proposition: $1 + a + a^2 + \ldots + a^n$

$$= (a^{n+1} - 1) / (a - 1)$$

Base step: $p(0)$ is $1 = (a^{0+1} - 1) / (a - 1)$ which is true

$$1 = (a - 1) / (a - 1)$$

$$1 = 1$$

Example 3 (Cont.):

Induction step: Assume $p(n)$ is true, i.e.,
$1 + a + a^2 + \ldots + a^n = (a^{n+1} - 1) / (a - 1)$

We need to show $p(n + 1)$: $1 + a + a^2 + \ldots + a^{n+1} = (a^{n+2} - 1) / (a - 1)$

That is $p(n + 1)$: $1 + a + a^2 + \ldots + a^n + a^{n+1} = (a^{n+2} - 1) / (a - 1)$

LHS of $p(n + 1) = (1 + a + a^2 + \ldots + a^n) + a^{n+1}$
$= (a^{n+1} - 1) / (a - 1) + a^{n+1}$
$= 1 / (a - 1) [a^{n+1} - 1 + (a - 1) a^{n+1}]$
$= 1 / (a - 1) [a^{n+1} - 1 + a^{n+2} - a^{n+1})]$
$= (a^{n+2} - 1) / (a - 1) =$ RHS of $p(n + 1)$

Thus, $p(n + 1)$: $1 + a + a^2 + \ldots + a^{n+1} = (a^{n+2} - 1) / (a - 1)$ is true

A special case of the above sum is when summing the first n powers of 2: $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$.

Induction Proof – Example 4

Use induction to show that $n < 2^n$ for all integers $n \geq 1$.

Let $p(n)$ be the proposition:
$n < 2^n$ for $n \geq 1$ where n is integer

Base step: $p(1)$ is true because $1 < 2^1$.

Induction step: Assume $p(k)$ is true, i.e., $k < 2^k$

We need to show that $p(k + 1)$ is true, that is $k + 1 < 2^{k+1}$

$k < 2^k \rightarrow k + 1 < 2^k + 1$ and $2^k + 1 \leq 2^k + 2^k = 2^{k+1}$ --- $(1 \leq 2^k)$

Thus $k + 1 < 2^{k+1}$ which means that $p(k + 1)$ is true.

Ex: 5: Use induction to show that $2^n < n!$ for $n \geq 4$.

Let $p(n)$ be the proposition, $2^n < n!$ for $n \geq 4$.

Base step: $p(4)$ is true as $2^4 = 16 < 4! = 24$

Induction step: Assume $p(k)$ is true, i.e., $2^k < k!$ for $k \geq 4$.

We need to show that $p(k + 1)$ is true: $2^{k+1} < (k + 1)!$ for $k \geq 4$.

$2^{k+1} = 2 \,.\, 2^k < 2\, k! < (k + 1)\, k! = (k + 1)!$ for $k \geq 4$ $(2 < (k + 1))$.

This shows $p(k + 1)$ is true when $p(k)$ is true.

We have completed the base and induction steps; thus, we have shown that $p(n)$ is true for $n \geq 4$.

Ex 6: Prove that $n^3 - n$ is divisible by 3 for all integers $n \geq 1$.

Let $p(n)$ be the proposition that $n^3 - n$ is divisible by 3.

Base step: $p(1)$ is true as $1^3 - 1 = 0$ which is divisible by 3.

Induction step: Assume $p(k)$ is true, that is, $k^3 - k$ is divisible by 3.

We must show that $p(k + 1)$ is true, that is,
$(k + 1)^3 - (k + 1)$ is divisible by 3.

$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1) =$

$= k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 3k - k = (k^3 - k) + 3(k^2 + k)$

The term $(k^3 - k)$ is divisible by 3 by the induction hypothesis, while the term $3(k^2 + k)$ is clearly divisible by 3 (it is a multiple of 3). Thus, $(k^3 - k) + 3(k^2 + k)$ is divisible by 3.
Thus, $(k + 1)^3 - (k + 1)$ is divisible by 3.

Ex 7

Prove that if S is a finite set with n elements, then S has 2^n subsets.

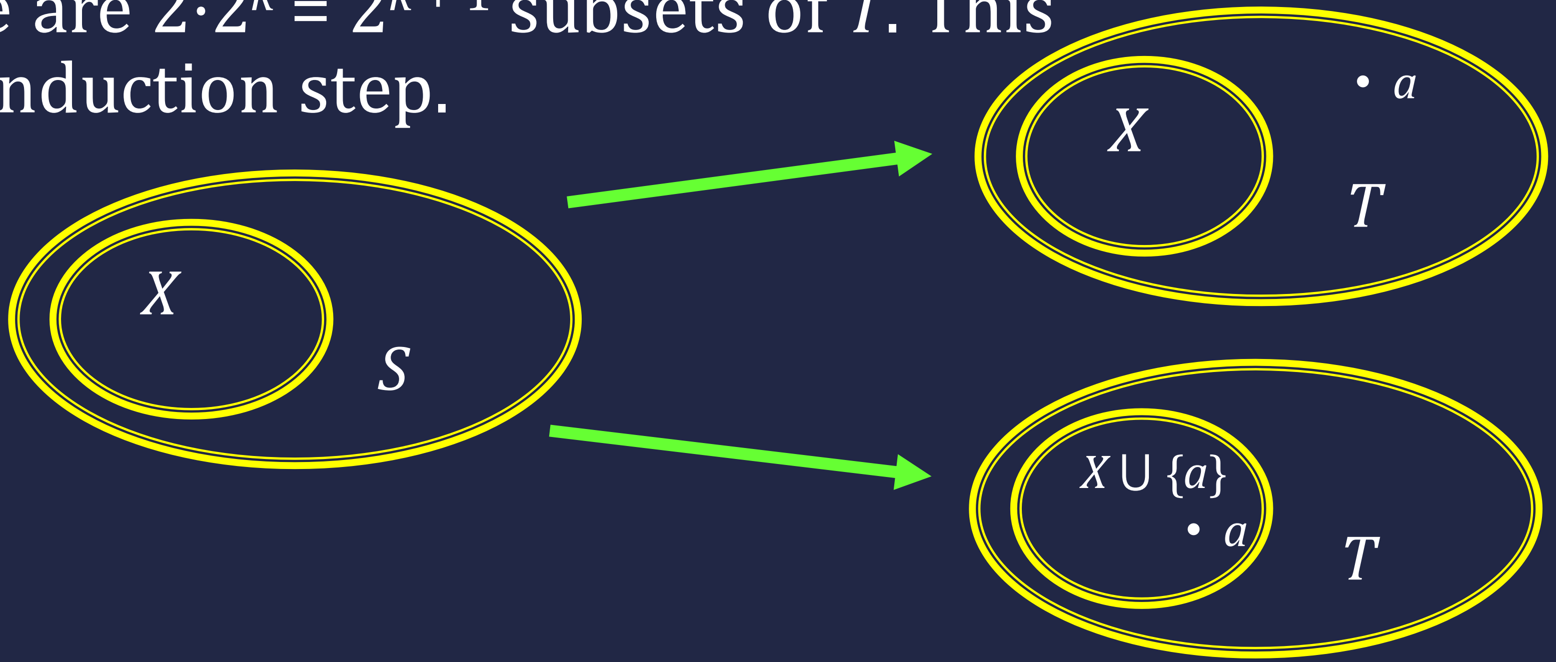
Let $p(n)$ be the proposition that a set with n elements has 2^n subsets.

Base step: $p(0)$ is true as a set with zero elements, the empty set, has exactly $2^0 = 1$ subset.

Induction step: Assume $p(k)$ is true, i.e., S has 2^k subsets if $|S|=k$.

Induction Proof – Example 7 (Cont.)

- Let T be a set with $k + 1$ elements such that $T = S \cup \{a\}$, where $a \notin S$.
- Each subset X of S corresponds to exactly two subsets of T, namely: X and $X \cup \{a\}$.
- By the induction hypothesis, there are 2^k subsets of S. Therefore there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This completes the induction step.



Template for *Proofs by Mathematical Induction*

1. Express the statement that is to be proved in the form “for all $n \geq b$, $P(n)$” for a fixed integer b.
2. Write out the words “Basis Step.” Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words “Inductive Step.”
4. State, and clearly identify, the inductive hypothesis, in the form “assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$.”
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k + 1)$ says.
6. Prove the statement $P(k + 1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying “this completes the inductive step.”
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.