5.2 Strong Induction

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Strong induction

- In strong induction, for the induction step, we use a stronger induction hypothesis than (weak) induction.
- To prove $p(n)$ for all $n \geq 1$: Instead of assuming $p(k)$, you can assume the statements, $p(1)$, $p(2)$, ..., $p(k)$ to prove $p(k+1)$.
- Weak induction and strong induction are equivalent.
- Any proof using weak induction can also be considered to be a proof by strong induction
- It is sometimes difficult to convert a proof by strong induction to one that uses weak induction.

Strong induction - Example 1

Use strong induction to show that if n is an integer greater than 1, then n can be written as a product of (one or more) primes.

Let $p(n)$ be the proposition that n can be written as a product of primes.

Base step: $p(2)$ is true (why?)

Induction hypothesis: Assume $p(j)$ is true for $2 \le j \le k$; that is, j $(2 \le j \le k)$ can be written as a product of primes.

Strong induction - Example 1 (cont.)

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Let $p(n)$ be the proposition that n can be written as a product of primes.

Base step: $p(2)$ is true

Induction hypothesis: Assume $p(j)$ is true for $2 \leq j \leq k$; that is, j $(2 \leq j \leq k)$ can be written as a product of primes.

To complete the proof, we need to show $p(k + 1)$ is true

(i.e., $k + 1$ can be written as a product of primes)

There are two cases: $k + 1$ is either prime or composite

If $k + 1$ is prime, we immediately see that $p(k + 1)$ is true

If $k + 1$ is composite then $k + 1 = ab$ for some two positive integers a and b, with $2 \leq a \leq b \leq k$; *by the inductive hypothesis, both a and b can be written as product of primes*

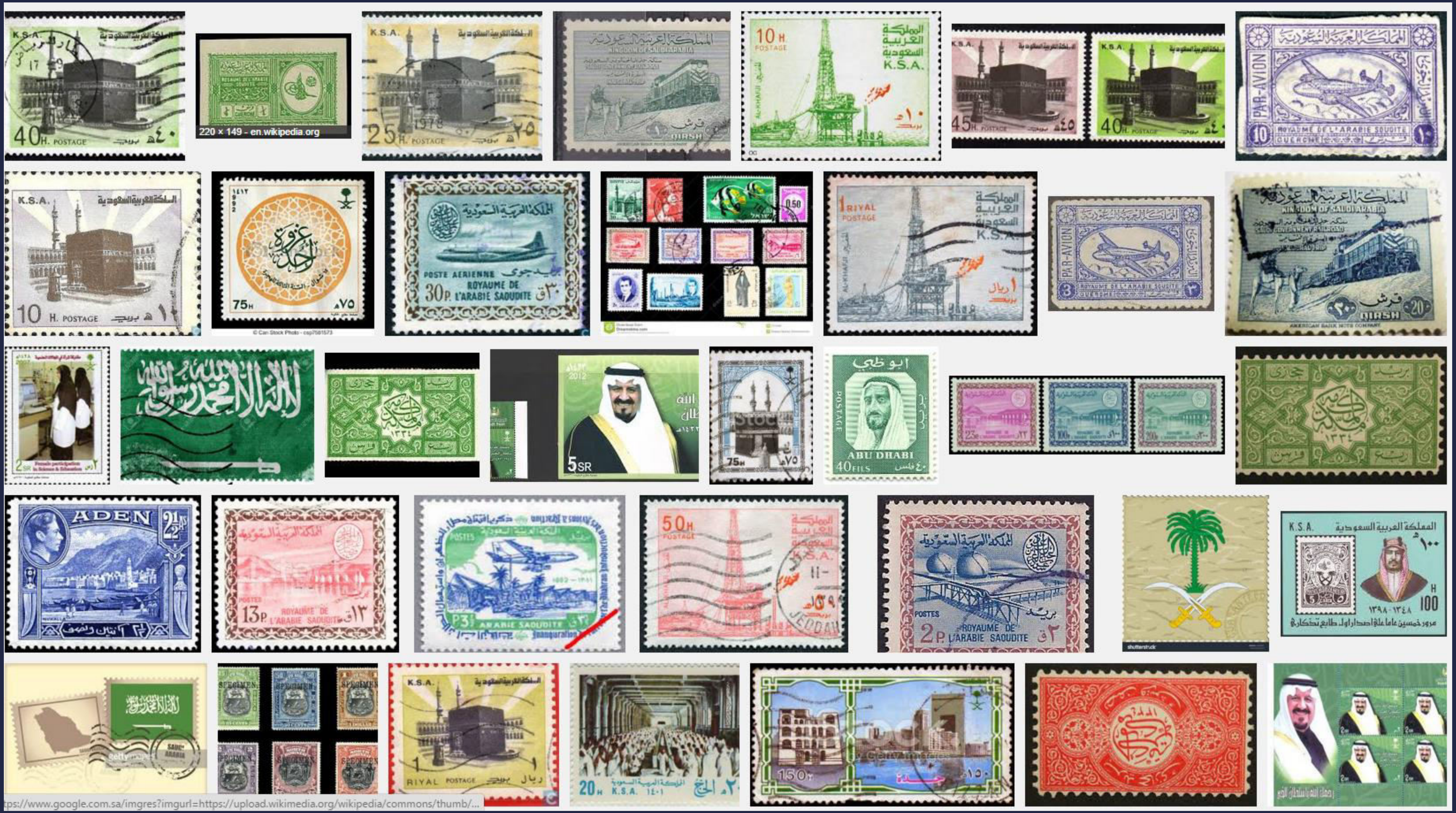
So $k + 1$ can be written as a product of primes

Example 2

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Think of Cents of the Dollars as the Halals of the Riyal.

Postage stamps



Example 2

Solution 1: Weak Induction

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Base step: A postage of 12 cents can be formed using three 4-cent stamps.

Induction step: We must show that for all $k \geq 12$, $p(k) \rightarrow p(k+1)$.

Thus, we assume $p(k)$ is true and show $p(k+1)$ is true.

Example 2 – Solution 1 – Cont.

Suppose that at least one 4-cent stamp is used for the postage of k cents.

In this case, we can replace one 4-cent stamp with one 5-cent stamp to get a postage of $k + 1$ cents.

On the other hand, if no 4-cent stamps are used, then, because $k \geq 12$,

there are at least three 5-cent stamps.

In this case, we can replace three 5-cent stamps with four 4-cent stamps for $k + 1$ cents.

Example 2 Solution 2: Strong Induction

As a base step, we show that $p(12)$, $p(13)$, $p(14)$ and $p(15)$ are true.

In the induction step, we show that how to get a postage of $k + 1$ cents for $k \geq 15$ from a postage of $k - 3$ cents.

Base step: we can form a postage of 12, 13, 14, 15 cents as follow:

Postage of 12 cents: three 4-cent

Postage of 13 cents: two 4-cent + one 5-cent

Postage of 14 cents: two 5-cent + one 4-cent

Postage of 15 cents: three 5-cent.

Thus $p(12)$, $p(13)$, $p(14)$, $p(15)$ are true.

Example 2 – Solution 2 – Cont.

Induction step: The induction hypothesis is the statement $p(j)$ is true for $12 \le j \le k$, where k is an integer with $k \ge 15$.
We need to show $p(k + 1)$ is true.

We can assume $p(k - 3)$ is true because $k - 3 \ge 12$, that is, we can form a postage of $k - 3$ cents using just 4-cent and 5-cent stamps.

To form postage of $k + 1$ cents, we simply add one 4-cent stamp to the postage of $k - 3$ cents. That is, we show $p(k + 1)$ is true.

we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that $P(n)$ is true for all integers n with $n \ge 12$. That is, we know that every postage of n cents, where n is at least 12, can be formed using 4-cent and 5-cent stamps. This finishes the proof by strong induction.

Example 2 – Solution 2 – Cont.

Note: In a strong induction proof where the induction step expresses $P(n + 1)$ in terms of $P(n - m)$, the base step must be established for

$$m + 1 \text{ values: } n_0, n_0 + 1, \ldots, n_0 + m.$$

(Note: $m = 0$ corresponds to *weak induction*)