1.6 Rules of Inference

Credit

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1.6 Rules of Inference

- **Proof**: valid arguments that establish the truth of a mathematical statement
- **Argument**: a sequence of statements that end with a conclusion
- **Valid**: the conclusion or final statement of the argument must follow the truth of preceding statements or **premises** of the argument

Argument and inference

- An **argument** is valid *if and only if* it is *impossible* for all the premises to be *true* and the conclusion to be *false*
- Rules of **inference**: use them to deduce (construct) new statements from statements that we already have
  - Basic tools for establishing the truth of statements

Valid arguments in propositional logic

- Consider the following arguments involving propositions

p

$p \rightarrow q$ "If you have a current password, then you can login to the network" q

p "You have a current password"

premises

$\therefore$ therefore,

q "You can login to the network" conclusion

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Valid arguments

p
$p \to q$

$\therefore q$

- $(p \land (p \to q)) \to q$ is tautology
- When $(p \land (p \to q))$ is true, both p and $p \to q$ are true, and thus q must be also be true
- This form of argument is true because when the premises are true, the conclusion must be true

Valid arguments

p
$p \to q$
___
$\therefore q$

$(p \wedge (p \to q)) \to q$ is is a tautology (always true)

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

$\therefore$ therefore

This is another way of saying that

What happens when we replace p and q in this argument form by propositions where not both p and $p \rightarrow q$ are true?

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

- p: "You have access to the network"
- q: "You can change your grade"
- $p \rightarrow q$: "If you have access to the network, then you can change your grade" (Can you?)

"If you have access to the network, then you can change your grade" ($p \rightarrow q$)) ($p \rightarrow q$ is FALSE

"You have access to the network" (p) (Given))

so "You can change your grade" (q)

p
$p \rightarrow q$
———
$\therefore q$

What happens when we replace p and q in this argument form by propositions where not both p and $p \rightarrow q$ are true?

"If you have access to the network, then you can change your grade" ($p \rightarrow q$)

"You have access to the network" (p)

so "You can change your grade" (q)

- Valid arguments
- But the conclusion is not true because ($p \rightarrow q$) is false
  - we cannot conclude that the conclusion is true.
- **Argument form**: a sequence of compound propositions involving propositional variables

Rules of inference for propositional logic

- Can always use truth table to show an argument form is valid
- For an argument form with 10 propositional variables, the truth table requires 2^{10} rows (1024)
- The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the rule of inference called **modus ponens** (*mode that affirms*), or the **law of detachment**

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Argument form: a sequence of compound propositions involving propositional variables

Example

- If both statements
  - "If it snows today, then we will go skiing" $p \to q$
  - "It is snowing today" p

are true

- By modus ponens,
  - it follows the conclusion
    - "We will go skiing" q

      is true

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

Example

- is the following argument valid? Is the conclusion true?

  If $\sqrt{2} > \frac{3}{2}$, *then* $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. ~~We know that $\sqrt{2} > \frac{3}{2}$.~~

  Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

- Let p be $\sqrt{2} > \frac{3}{2}$. Let q be $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$
  - The premises of the argument are $p \rightarrow q$ and p,
  - the conclusion is q
- This argument is valid by using modus ponens
- But one of the premises ($\sqrt{2} > \frac{3}{2}$) is false, consequently we cannot conclude the conclusion is true
- The conclusion is false.

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Rules of Inference

- Means to draw conclusions from other assertions
- Rules of inference provide justification of steps used to show that a conclusion follows from a set of hypotheses (premises)
- The next several slides illustrate specific rules of inference

Addition

A true hypothesis implies that the disjunction of that hypothesis and another are true

$$\frac{p}{\therefore\; p \vee q}$$

or $p \rightarrow (p \vee q)$

Simplification

If the conjunction of 2 propositions is true, then each proposition is true

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array} \qquad \text{or} \quad (p \wedge q) \rightarrow p$$

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ q \end{array} \qquad \text{Also} \quad (p \wedge q) \rightarrow q$$

Conjunction

If p is true and q is true, then $p \wedge q$ is true

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

or $((p) \wedge (q)) \rightarrow p \wedge q$

Modus Ponens

If a hypothesis and implication are both true, then the conclusion is true

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array} \qquad \text{or} \qquad (p \wedge (p \to q)) \to q$$

Modus Tollens

If a conclusion is false and its implication is true, then the hypothesis must be false

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

or $(\neg q \wedge (p \to q)) \to \neg p$

Hypothetical Syllogism

If an implication is true, and the implication formed using its conclusion as the hypothesis is also true, then the implication formed using the original hypothesis and the new conclusion is also true

$$
\begin{array}{l}
p \rightarrow q \\
q \rightarrow r \\
\hline
\therefore\ p \rightarrow r
\end{array}
$$

or $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Disjunctive Syllogism

If a proposition is false, and the disjunction of it and another proposition is true, the second proposition is true

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore\ q \end{array}$$

or, $((p \lor q) \land \neg p) \rightarrow q$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

or, $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

$$\begin{array}{l} p \vee q \\ \neg p \vee q \\ \hline \therefore\ q \vee q = \boldsymbol{q} \end{array} \qquad \begin{array}{l} p \\ \neg p \vee q \\ \hline \therefore\ q \end{array}$$

What do we obtain if we let $q = r$?

$$\begin{array}{l} p \vee q \\ \neg p \vee \mathbf{F} \\ \hline \therefore\ q \vee \mathbf{F} = \boldsymbol{q} \end{array} \qquad \begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$$

What do we obtain if we let $r = \mathbf{F}$?

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Example

Let p = "It is Sunday" and

$p \rightarrow q$ = "If it is Sunday, I have Physics Lab today"

If these statements are both true,

then by Modus Ponens:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

we can conclude "I have Physics Lab today" (q)

p

$p \rightarrow q$

$\therefore q$

Example

Let $\neg q$ = "I don't have Physics Lab today" and
$p \rightarrow q$ = "If it is Sunday, I have Physics Lab today"
If both of the above are true, then by Modus Tollens:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

we can conclude "It is not Sunday" ($\neg p$)

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Example of proof

- We have the hypotheses (premises):
  - “It is not sunny this afternoon and it is colder than yesterday”
  - “We will go swimming only if it is sunny”
  - “If we do not go swimming, then we will take a canoe trip”
  - “If we take a canoe trip, then we will be home by sunset”
- Does this imply that “we will be home by sunset”?

Solution

- We have the hypotheses:
  - p, q — "It is not sunny this afternoon and it is colder than yesterday"
  - r — "We will go swimming only if it is sunny"
  - s — "If we do not go swimming, then we will take a canoe trip"
  - u — "If we take a canoe trip, then we will be home by sunset"
- Does this imply that "we will be home by sunset"?

hypotheses (premises)

$\neg p \wedge q$

$r \rightarrow p$

$\neg r \rightarrow s$

$s \rightarrow u$

Conclusion

u ?

Solution (continued)

1.	$\neg p \wedge q$	1st hypothesis (premise)
2.	$\neg p$	Simplification using step 1
3.	$r \rightarrow p$	2nd hypothesis
4.	$\neg r$	Modus tollens using steps 2 & 3
5.	$\neg r \rightarrow s$	3rd hypothesis
6.	s	Modus ponens using steps 4 & 5
7.	$s \rightarrow u$	4th hypothesis
8.	u	Modus ponens using steps 6 & 7

Premises

$\neg p \wedge q$

$r \rightarrow p$

$\neg r \rightarrow s$

$s \rightarrow u$

$$\frac{p \wedge q}{\therefore p}$$

$$\frac{\begin{array}{l} p \\ p \rightarrow q \end{array}}{\therefore q}$$

$$\frac{\begin{array}{l} \neg q \\ p \rightarrow q \end{array}}{\therefore \neg p}$$

So what did we show?

- We showed that:
  - $((\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to u)) \to u$
  - That when the 4 hypotheses are true, then the implication is true
  - In other words, we showed the above is a tautology!

hypotheses (premises)

$\neg p \wedge q$

$r \to p$

$\neg r \to s$

$s \to u$

Conclusion

u

So what did we show?

- To show this, enter the following into the truth table generator at

  http://turner.faculty.swau.edu/mathematics/materialslibrary/truth/

  $$((\sim P \,\&\, Q) \,\&\, (R > P) \,\&\, (\sim R > S) \,\&\, (S > U)) > U$$

  $$((\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow u)) \rightarrow u$$

- Do you want to search for other truth table generators?

Example

- Show that the premises (hypotheses)
  - "If you send me an email message, then I will finish my program"
  - "If you do not send me an email message, then I will go to sleep early"
  - "If I go to sleep early, then I will wake up feeling refreshed"
- lead to the conclusion
  - "If I do not finish writing the program, then I will wake up feeling refreshed"

Solution

- Let
  - p “You send me an e-mail message”
  - q “I will finish writing the program”
  - r “I will go to sleep early”
  - s “I will wake up feeling refreshed”

Solution

- Show that the premises
  - "If you send me an email message, then I will finish my program"
  - "If you do not send me an email message, then I will go to sleep early"
  - "If I go to sleep early, then I will wake up feeling refreshed"
- lead to the conclusion
  - "If I do not finish writing the program, then I will wake up feeling refreshed"

- *Premises (Hypotheses)*
  - $p \rightarrow q$
  - $\neg p \rightarrow r$
  - $r \rightarrow s$
- *Conclusion*
  - $\neg q \rightarrow s$

- p "You send me an e-mail message"
- q "I will finish writing the program"

- r "I will go to sleep early"
- s "I will wake up feeling refreshed"

So given $p \to q,\ \neg p \to r, r \to s$ show $\neg q \to s$

Step	**Reason**
[1] $p \to q$	Premise
[2] $\neg q \to \neg p$	Contrapositive of (1)
[3] $\neg p \to r$	Premise
[4] $\neg q \to r$	Hypothetical syllogism using (2) and (3)
[5] $r \to s$	Premise
[6] $\neg q \to s$	Hypothetical syllogism using (4) and (5)

- p "You send me an e-mail message"
- q "I will finish writing the program"

- r "I will go to sleep early"
- s "I will wake up feeling refreshed"

Special case of Resolution

- Based on the tautology $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
  - Resolvent: $(q \vee r)$
- Let $r = q$, we have $((p \vee q) \wedge (\neg p \vee q)) \rightarrow q$
- Let $r = \mathrm{F}$, we have $((p \vee q) \wedge \neg p) \rightarrow q$
- Important in logic programming, AI, etc.

$p \vee q$	$p \vee q$	$p \vee q$
$\neg p \vee r$	$\neg p \vee q$	$\neg p \vee \mathbf{F}$
$\therefore\ q \vee r$	$\therefore\ q$	$\therefore\ q$

Example

- "It is hot or Sami is reading" $p \vee q$
- "It is not hot or Ali is swimming " $\neg p \vee r$
- imply $\rightarrow$
- "Ali is swimming or Sami is reading" $q \vee r$

- $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$ (Resolution)

Example

- To construct proofs using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as clauses
- **Clause**: a disjunction of variables or negations of these variables
- To show that $(p \wedge q) \vee r$ and $r \rightarrow s$ imply $p \vee s$

We start by showing

- $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ Expressed as two clauses
- $r \rightarrow s \equiv \neg r \vee s$ Expressed as a clause

Then we continue

Fallacies

fallacy of affirming the conclusion

- **Inaccurate** arguments
- Is $((p \rightarrow q) \wedge q) \rightarrow p$ a tautology? Find out!
- $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology as it is **FALSE** when p is **FALSE** and q is **TRUE**
- p If you do every problem in th
  then you will learn dis q
- q You learned discrete mathem

  Therefore you did every prob

$(p \rightarrow q) \wedge q$

Fallacy

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$((p \rightarrow q) \wedge q) \rightarrow p$
F	F	T	F	T
F	T	T	T	F
T	F	F	F	T
T	T	T	T	T

Not a tautology

Example

- $((p \to q) \wedge \neg p)$ is it correct to conclude $\neg q$?
- Is $((p \to q) \wedge \neg p) \to \neg q$ a tautology? Not a tautology

$(p \to q) \equiv \neg q \to \neg p$

- Fallacy: the incorrect argument is of the form as $\neg p$ does not imply $\neg q$
- $(p \to q)$ is not equivalent to

fallacy of denying the hypothesis

$\neg p \to \neg q$

it is false when p is false and q is true

Rules of Inference for Quantified Statements

- **Universal instantiation:**

$$\frac{\forall x\, P(x)}{\therefore\ P(c) \text{ if } c \in \mathrm{U}}$$

(U is the domain (Universe of Discourse))

- **Universal generalization:**

$$\frac{P(c) \text{ for arbitrary } c \in \mathrm{U}}{\therefore\ \forall x\, P(x)}$$

(U is the domain (Universe of Discourse))

Note: c must be arbitrary

Rules of Inference for Quantified Statements

- **Existential instantiation:**

  $\exists x\, P(x)$

  ____

  $\therefore\ P(c)$ for some $c \in U$ (U is the domain (Universe of Discourse))

  Note that value of c is not known; we only know it exists

- **Existential generalization:**

  $P(c)$ for some $c \in U$

  ____

  $\therefore\ \exists x\, P(x)$

Inference with quantified statements

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$	Existential generalization

Instantiation:
c is one *particular* member of the domain

Generalization:
for an *arbitrary* member *c*

Example

Let $P(x)$ = "A man is mortal"; then

$\forall x\, P(x)$ = "All men are mortal"

Assuming r = "Ali is a man" is true, show that
s = "Ali is mortal" is implied

This is an example of universal instantiation:

$P(\text{Ali})$ = "Ali is mortal";

$$\frac{\text{Since } \forall x\, P(x)}{\therefore\ P(c)}$$

Example

- Show that "Everyone in this discrete mathematics class has taken a course in computer science" and "Musab is a student in this class" imply "Musab has taken a course in computer science"
- Assume $D(x)$ represents x is in Discrete mathematics class
- Assume $C(x)$ represents x has taken a course in computer Science

Step	**Reason**
[1] $\forall x\,(D(x) \rightarrow C(x))$	Premise
[2] $D(\text{Musab}) \rightarrow C(\text{Musab})$	Universal instantiation from (1)
[3] $D(\text{Musab})$	Premise
[4] $C(\text{Musab})$	Modus ponens from (2) and (3)

Example

Show that

"A student in this class has not read the book", and
"Everyone in this class passed the first exam" imply
"Someone who passed the first exam has not read the book"

- Assume $C(x)$ represents x is a student in the class
- Assume $B(x)$ represents x has read the book
- Assume $P(x)$ represents x has passed the first exam

Example

Step	**Reason**
[1] $\exists x\,(C(x) \land \neg B(x))$	Premise
[2] $C(a) \land \neg B(a)$	Existential instantiation from [1]
[3] $C(a)$	Simplification from [2]
[4] $\forall x\,(C(x) \rightarrow P(x))$	Premise
[5] $C(a) \rightarrow P(a)$	Universal instantiation from [4]
[6] $P(a)$	Modus ponens from [3] and [5]
[7] $\neg B(a)$	Simplification from [2]
[8] $P(a) \land \neg B(a)$	Conjunction from [6] and [7]
[9] $\exists x\,(P(x) \land \neg B(x))$	Existential generalization from [8]

Show that "A student in this class has not read the book", and "Everyone in this class passed the first exam" imply "Someone who passed the first exam has not read the book"

$C(x)$: x is a student in the class. $B(x)$: x has read the book. $P(x)$: x has passed the first exam

Universal modus ponens

- Use universal instantiation and modus ponens to derive new rule

$\forall x\,(P(x) \rightarrow Q(x))$

$\underline{P(a), \text{ where } a \text{ is a particular element in the domain}}$

$\therefore Q(a)$

Example: Universal modus ponens

- Assume "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Show $100^2 < 2^{100}$.

$100^2 = 10000$

$< 2^{100} = 1267650600228229401496703205376$

- *Solution:*
- Let $P(n)$ be "$n > 4$" and $Q(n)$ be "$n^2 < 2^n$"
- The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$
- where the domain consists of all positive integers.
- We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true.
- Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$

Universal modus tollens

- Combine universal modus tollens and universal instantiation

$\forall x\,(P(x) \rightarrow Q(x))$

$\underline{\neg Q(a)\text{, where } a \text{ is a particular element in the domain}}$

$\therefore\ \neg P(a)$