The Fundamentals of Logic
Propositional Equivalences

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Propositional Equivalences

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Propositional Satisfiability
- Sudoku Puzzle

Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is **always true**.

  Example: $p \vee \neg p$

- A **contradiction** is a proposition which is **always false**.

  Example: $p \wedge \neg p$

- A **contingency** is a proposition which is neither a tautology nor a contradiction

Example: p

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
F	T	T	F
T	F	T	F

Logically Equivalent

- Two compound propositions r and s are logically **equivalent** if $r \leftrightarrow s$ **is a tautology**.
- Denoted by $r \Leftrightarrow s$ or as $r \equiv s$ where r and s are compound propositions.
- In other words, two compound propositions r and s are equivalent **if and only if** the columns in a truth table giving their truth values **agree**.

Logically Equivalent

- The following truth table shows $\boldsymbol{r} : \neg p \lor q$ is equivalent to $\boldsymbol{s}$: $p \rightarrow q$. (and s is Equivalent to r)

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \neg p \vee \neg q \equiv \neg(p \wedge q)$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad \neg p \wedge \neg q \equiv \neg(p \vee q)$$

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

Key Logical Equivalences

- Identity Laws: $p \wedge \mathbf{T} \equiv p$ $\quad p \vee \mathbf{F} \equiv p$
  $p \equiv p \wedge \mathbf{T}$ $\quad p \equiv p \vee \mathbf{F}$
- Domination Laws: $p \vee \mathbf{T} \equiv \mathbf{T}$ $\quad p \wedge \mathbf{F} \equiv \mathbf{F}$
  $\mathbf{T} \equiv p \vee \mathbf{T}$ $\quad \mathbf{F} \equiv p \wedge \mathbf{F}$
- Idempotent laws: $p \vee p \equiv p$ $\quad p \wedge p \equiv p$
  $p \equiv p \vee p$ $\quad p \equiv p \wedge p$
- Double Negation Law: $\neg(\neg p) \equiv p$
  $p \equiv \neg(\neg p)$
- Negation Laws: $p \vee \neg p \equiv \mathbf{T}$ $\quad p \wedge \neg p \equiv \mathbf{F}$
  $\mathbf{T} \equiv p \vee \neg p$ $\quad \mathbf{F} \equiv p \wedge \neg p$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p$
  $p \wedge q \equiv q \wedge p$
- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws: $p \vee (p \wedge q) \equiv p$
  $p \wedge (p \vee q) \equiv p$

Logical equivalences involving conditional statement

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$\neg p \vee q \equiv p \to q$

$\neg q \to \neg p \equiv p \to q$

$\neg p \to q \equiv p \vee q$

$\neg(p \to \neg q) \equiv p \wedge q$

ABLE 7 Logical
nvolving Condition

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg$

$\neg(p \to q) \equiv p \wedge \neg$

$(p \to q) \wedge (p \to r$

$(p \to r) \wedge (q \to r)$

$(p \to q) \vee (p \to r$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$p \wedge \neg q \equiv \neg(p \to q)$

$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$

$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r)$

Logical equivalences involving biconditionals

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Review: List of Logical Equivalences

$p \wedge \mathbf{T} \equiv p$; $p \vee \mathbf{F} \equiv p$ Identity Laws

$p \vee \mathbf{T} \equiv \mathbf{T}$; $p \wedge \mathbf{F} \equiv \mathbf{F}$ Domination Laws

$p \vee p \equiv p$; $p \wedge p \equiv p$ Idempotent Laws

$\neg(\neg p) \equiv p$ Double Negation Law

$p \vee q \equiv q \vee p$; $p \wedge q \equiv q \wedge p$ Commutative Laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$; $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Associative Laws

List of Equivalences

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$		Distribution Laws
$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$		
$\neg(p \lor q)$	$\equiv (\neg p \land \neg q)$	De Morgan's Laws
$\neg(p \land q)$	$\equiv (\neg p \lor \neg q)$	
$p \lor \neg p$	$\equiv \mathbf{T}$	OR Tautology
$p \land \neg p$	$\equiv \mathbf{F}$	AND Contradiction
$(p \to q)$	$\equiv (\neg p \lor q)$	Implication Equivalence
$p \leftrightarrow q$	$\equiv (p \to q) \land (q \to p)$	Biconditional Equivalence

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$, we produce a series of equivalences beginning with A and ending with B.

$$A \equiv B$$

$$\begin{aligned} A &\equiv A_1 \\ A_1 &\equiv A_2 \\ &\vdots \\ A_n &\equiv B \end{aligned}$$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $(\neg p \wedge \neg q)$

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	2nd De Morgan L
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q)]$	1st De Morgan L
$\equiv \neg p \wedge (p \vee \neg q)$	Double Negation L
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	2nd Distribution L
$\equiv \mathrm{F} \vee (\neg p \wedge \neg q)$	$(\neg p \wedge p) \equiv \mathbf{F}$
$\equiv (\neg p \wedge \neg q) \vee \mathrm{F}$	Commutative L for Disjunction
$\equiv (\neg p \wedge \neg q)$	Identity L for F

Prove: $p \rightarrow q \equiv \neg q \rightarrow \neg p$ Contrapositive

$p \rightarrow q$

$\equiv \neg p \vee q$ Implication Equivalence

$\equiv q \vee \neg p$ Commutative

$\equiv \neg(\neg q) \vee \neg p$ Double Negation

$\equiv \neg q \rightarrow \neg p$ Implication Equivalence

Prove: $(p \land \neg q) \lor q \equiv p \lor q$

$(p \land \neg q) \lor q$	Left-Hand Statement
$\equiv q \lor (p \land \neg q)$	Commutative
$\equiv (q \lor p) \land (q \lor \neg q)$	Distributive
$\equiv (q \lor p) \land \mathbf{T}$	OR Tautology
$\equiv q \lor p$	Identity
$\equiv p \lor q$	Commutative

Begin with exactly the left-hand side statement
End with exactly what is on the right
Justify EVERY step with a logical equivalence

Prove: $\neg p \leftrightarrow q \equiv p \leftrightarrow \neg q$

$\neg p \leftrightarrow q$	
$\equiv (\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Biconditional Equivalence
$\equiv (\neg\neg p \vee q) \wedge (\neg q \vee \neg p)$	Implication Equivalence (twice)
$\equiv (p \vee q) \wedge (\neg q \vee \neg p)$	Double Negation
$\equiv (q \vee p) \wedge (\neg p \vee \neg q)$	Commutative twice
$\equiv (\neg\neg q \vee p) \wedge (\neg p \vee \neg q)$	Double Negation
$\equiv (\neg q \rightarrow p) \wedge (p \rightarrow \neg q)$	Implication Equivalence (twice)
$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$	Commutative
$\equiv p \leftrightarrow \neg q$	Biconditional Equivalence

Why do I have to justify everything?

- Note that your operation must have the same order of operands as the rule you quote unless you have already proven (and cite the proof) that order is not important.

  $3 + 4 = 4 + 3$

  $3 / 4 \neq 4 / 3$

  $A * B \neq B * A$ for everything (for example, matrix multiplication)

Propositional Satisfiability

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that make it **true**. When no such assignments exist, the compound proposition is **unsatisfiable**.

- A compound proposition is unsatisfiable **if and only if** its negation is a tautology.

Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p* and **F** to *q*.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Unsatisfiable – Find out.

$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true. For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to *p*, *q*, and *r* makes *the compound proposition* true. Hence, it is unsatisfiable.

- **Notation:**
  - $\bigvee_{j=1}^{n} p_j$ is used for $p_1 \vee p_2 \vee \ldots \vee p_n$
  - $\bigwedge_{j=1}^{n} p_j$ is used for $p_1 \wedge p_2 \wedge \ldots \wedge p_n$

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Sudoku puzzle

- A **Sudoku puzzle** is represented by a 9 × 9 grid made up of nine 3 × 3 subgrids, known as **blocks**.
- For each puzzle, some of the 81 cells, called **givens**, are assigned one of the numbers 1, 2, . . . , 9, and the other cells are blank.
- The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3 × 3 blocks contains each of the nine possible numbers.
- Note that instead of using a 9 × 9 grid, Sudoku puzzles can be based on $n^2 \times n^2$ grids, for any positive integer n, with the $n^2 \times n^2$ grid made up of n^2 subgrids $(n \times n)$.

	9			6	2			7
		4			3		8	6
	8	1	9			2		
1	5	8	3	4				
	7						9	
				8	6	5	1	3
		5			7	3	2	
9	6		4			8		
4			2	1			5	

Easy 4 by 4 puzzle

4	3	1	2
2	1	3	4
3	2	4	1
1	4	2	3

Easy 4 by 4 puzzle

3	1	4	2
4	2	3	1
1	3	2	4
2	4	1	3

Easy 4 by 4 puzzle

4	1	2	3
2	3	1	4
3	2	4	1
1	4	3	2

Easy 4 by 4 puzzle

3	1	4	2
4	2	3	1
2	3	1	4
1	4	2	3

4 by 4 puzzle

1	3	4	2
2	4	3	1
4	1	2	3
3	2	1	4

4 by 4 puzzle

2	3	4	1
4	1	3	2
3	2	1	4
1	4	2	3

4 by 4 puzzle

3	1	4	2
4	2	3	1
1	3	2	4
2	4	1	3

4 by 4 puzzle

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

9 by 9 puzzle

	9			6	2			7
		4			3		8	6
	8	1	9			2		
1	5	8	3	4	9			
	7						9	
		9		8	6	5	1	3
		5		9	7	3	2	
9	6		4			8		
4			2	1			5	

9 by 9 puzzle

5	9	3	8	6	2	1	4	7
7	2	4	1	5	3	9	8	6
6	8	1	9	7	4	2	3	5
1	5	8	3	4	9	7	6	2
3	7	6	5	2	1	4	9	8
2	4	9	7	8	6	5	1	3
8	1	5	6	9	7	3	2	4
9	6	2	4	3	5	8	7	1
4	3	7	2	1	8	6	5	9

Sudoku and logic?

- Let $p(i, j, n)$ denote the proposition asserting that the cell in row i and column j has the value n

$p(1, 3, 1)$

$p(1, 4, 2)$

$p(2, 1, 2)$

$p(4, 4, 3)$

$p(1, 1, n_1)$

$p(1, 2, n_2)$

$p(1, 3, n_3)$

$p(1, 4, n_4)$

$n_1 + n_2 + n_3 + n_4 = 10$

4	3	1	2
2	1	3	4
3	2	4	1
1	4	2	3