2.5 Cardinality of Sets

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Cardinality

Definition: the **cardinality** of a set A is equal to the cardinality of a set B, denoted by $|A| = |B|$ if and only if there is a one-to-one correspondence (bijection) from A to B.

- If there is a one-to-one function (injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

Definition: a set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$: {1, 2,...}) is called **countable**. A set that is not countable is **uncountable** (e.g., Real numbers).

Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence indexed by the positive integers $\mathbf{Z}^+$.

- The reason for this is that a one-to-one correspondence f (bijection) from the set of positive integers $\mathbf{Z}^+$ to a set $\boldsymbol{S}$ can be expressed in terms of a sequence as follows:

$$a_1, a_2, \ldots, a_n, \ldots \text{ where } a_1 = f(1),\ a_2 = f(2),\ \ldots,\ a_n = f(n),\ \ldots$$

Showing that a Set is Countable

Example: show that the set of positive even integers **E** is a countable set.

Solution: let $f(x) = 2x$.

$$
\begin{array}{llllllll}
\mathbf{Z^+}: & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbf{E}: & 2 & 4 & 6 & 8 & 10 & 12 & \dots
\end{array}
$$

Then f is a **bijection** from $\mathbf{Z^+}$ to **E** since f is both one-to-one and onto.

Positive Rational Numbers are Countable

- **Reminder**: a rational number can be expressed as the ratio of two integers p and q $\left(\frac{p}{q}\right)$ such that $q \neq 0$.
  - $\frac{3}{4}$ is a rational number
  - $\sqrt{2}$ is irrational (not rational)

Example: show that the positive rational numbers are countable.

Solution: the positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. $\rightarrow$

Positive Rational Numbers are Countable $\frac{p}{q}$

First row q = 1.

Second row q = 2

etc.

Fifth row q = 5

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{3}{5} & \frac{5}{5} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Positive Rational Numbers are Countable $\frac{p}{q}$

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\cdots$
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\cdots$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\cdots$
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\cdots$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Constructing the List

First list $\frac{p}{q}$ with $p + q = 2$

Second list $\frac{p}{q}$ with $p + q = 3$

Third list $\frac{p}{q}$ with $p + q = 4$, and so on.

Terms not circled are not listed because they were previously listed items

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

Countable Sets

- A set is countable if it is finite or
  if it can be placed in a 1-1 correspondence with the set of natural numbers,
  N = {1, 2, 3, …}.
- A countable set that is infinite has a cardinality of *aleph-null*. The symbol for *aleph-null* is $\aleph_0 :\equiv |\mathbf{N}|$

- The set $\mathbf{R}_{[0,1]}$ of reals between 0 and 1 is not countable.
- If A and B are countable sets, then $A \cup B$ is also countable.
- Any subset of a countable set is also countable

A Computable Function

- A function is ***computable*** if there is a computer program in some programming language that finds the values of this function
  - even with unlimited time and memory
- If a function is not computable we say it is ***uncomputable***.