6.2 Pigeonhole Principle

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Pigeonhole Principle

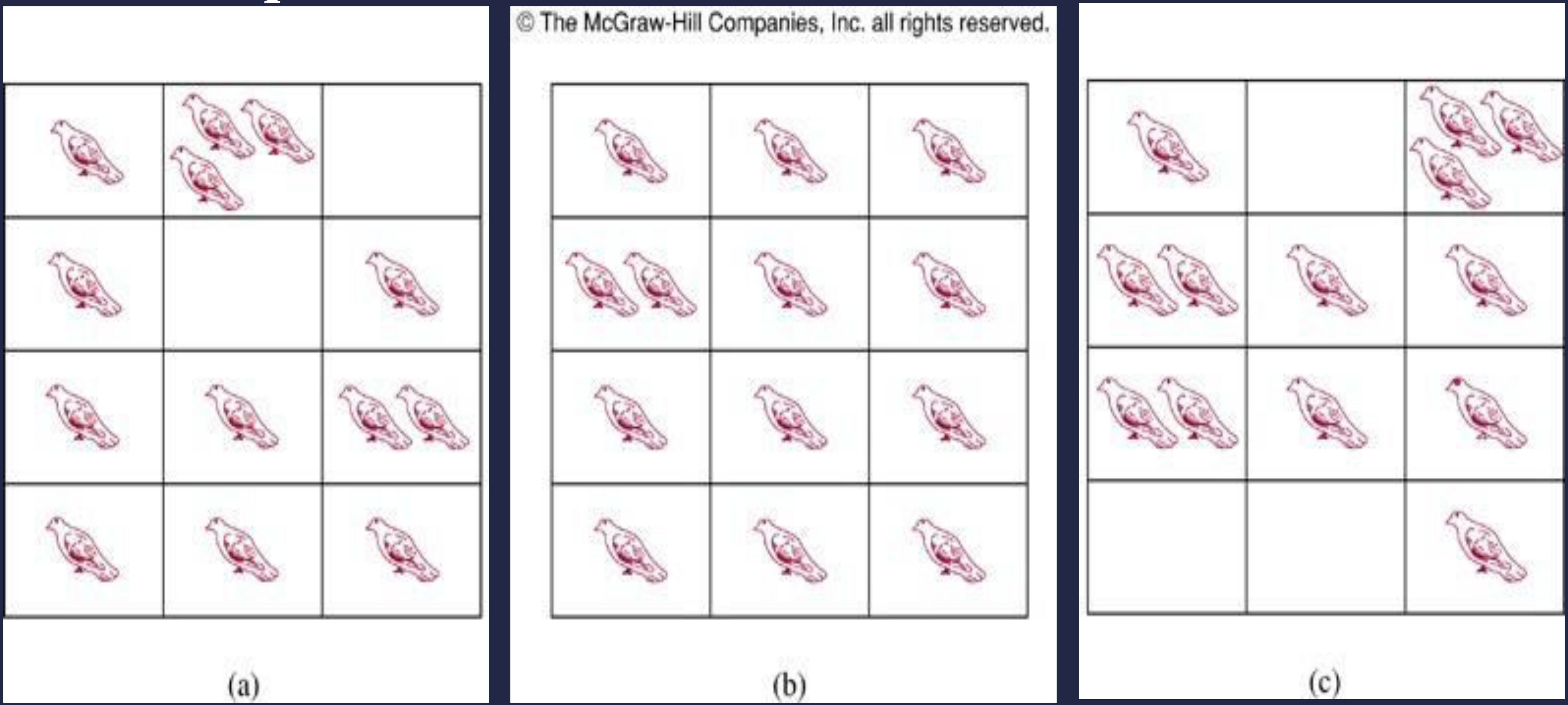
How many Pigeonholes?

25

How many Pigeons?

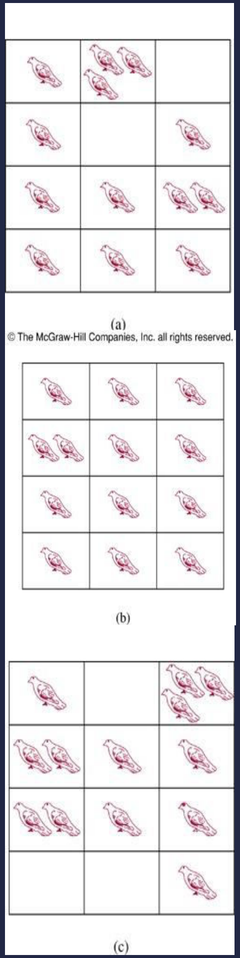
26

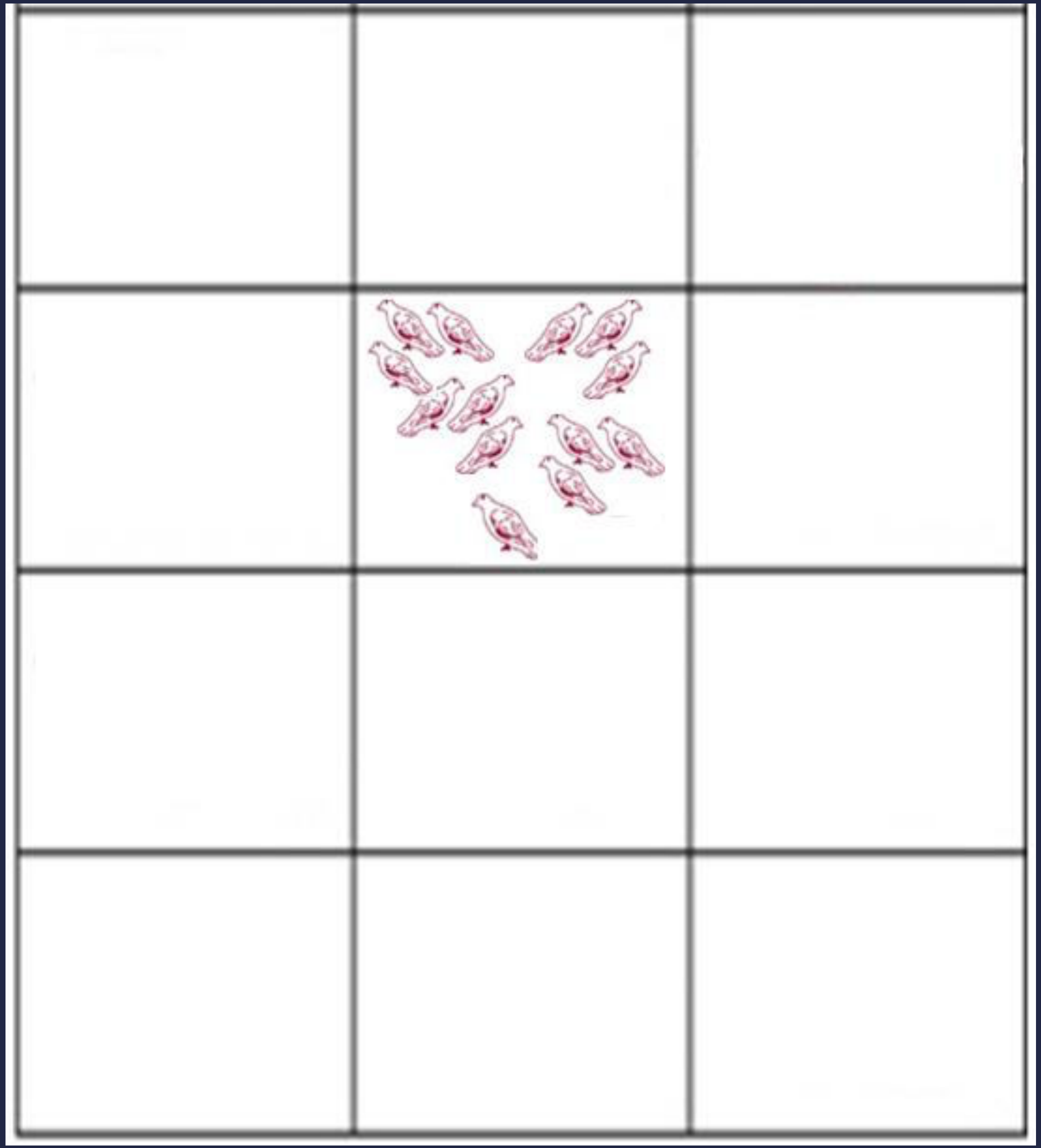
Example



13 pigeons and 12 pigeonholes

Example





13 pigeons and 12 pigeonholes

Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

Pigeonhole Principle: if k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two/more objects.

Pigeonhole Principle

pigeonhole principle:

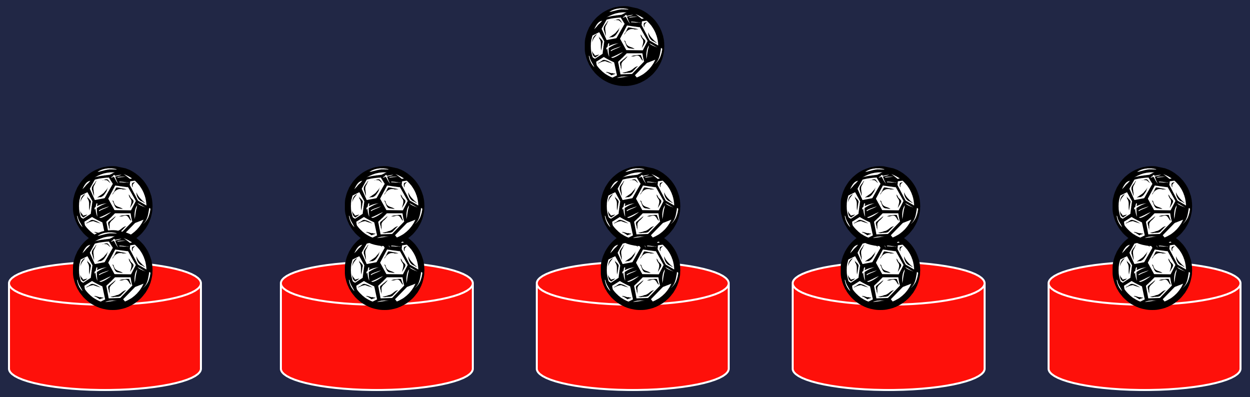
If we have $N > k$ balls and we divide them among k boxes, then at least one box contains at least 2 balls.

generalized pigeonhole principle:

If we have $N > k$ balls and we divide them among k boxes, then at least one box contains at least $\left\lceil \frac{N}{k} \right\rceil$ balls. Where $\left\lceil \frac{N}{k} \right\rceil$ is the ceiling value of $\frac{N}{k}$

$k = 5, N = 11$

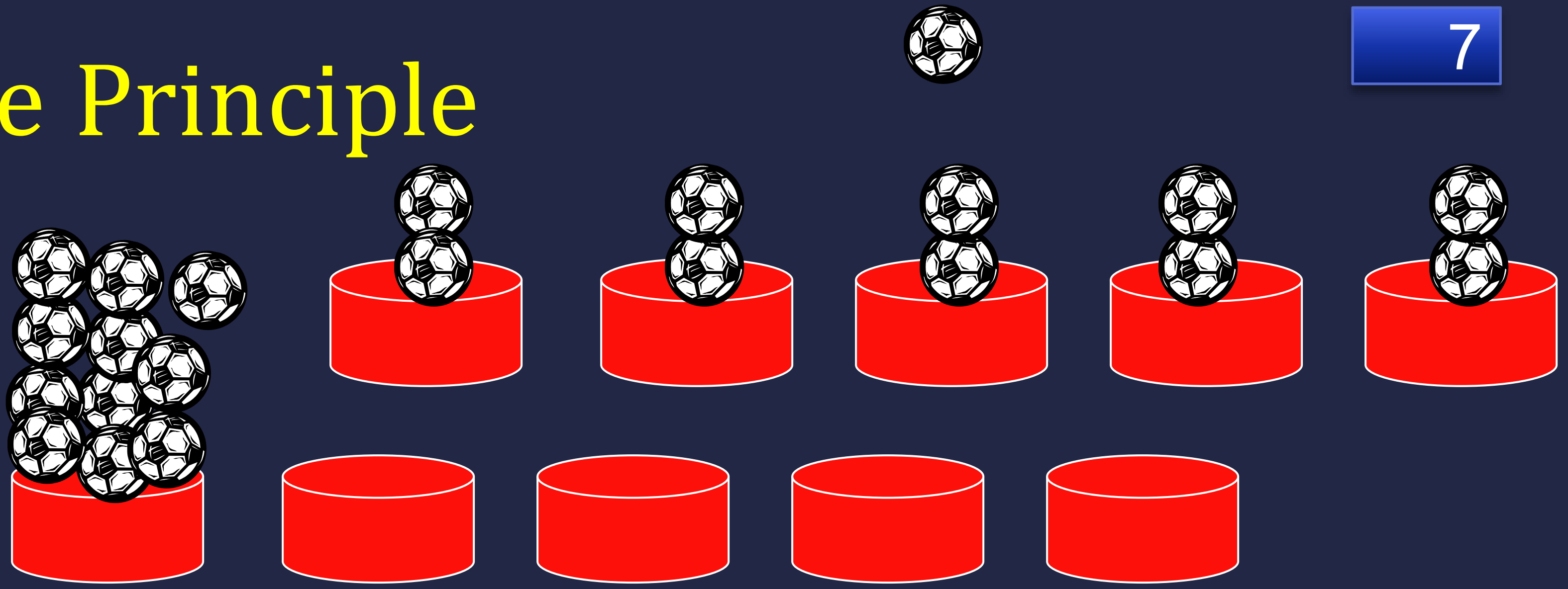
$\left\lceil \frac{N}{k} \right\rceil = 3$

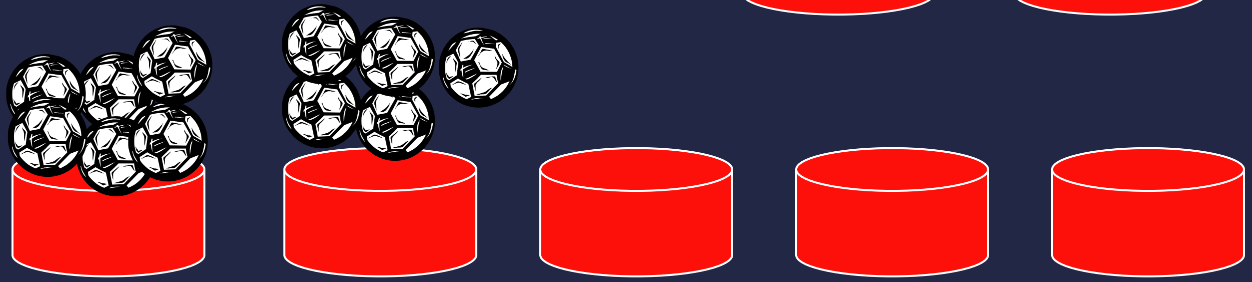


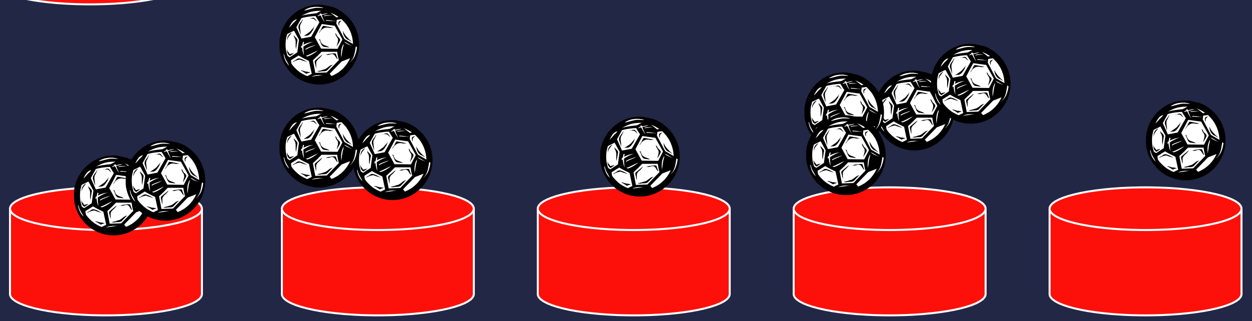
Pigeonhole Principle

$k = 5, N = 11$

$$\left\lceil \frac{N}{k} \right\rceil = 3$$







at least one box contains at least $\left\lceil \frac{N}{k} \right\rceil$ balls.

Pigeonhole Principle

Corollary: a function f from a set with $k+1$ elements to a set with k elements is not one-to-one.

Pigeonhole Principle

Example: among any group of 367 people, there must be at least two with the same birthday

because there are only 366 possible birthdays.

The Generalized Pigeonhole Principle: if N objects are placed into k boxes, then there is at least one box containing at least

$\left\lceil \frac{N}{k} \right\rceil$ objects. Where $\left\lceil \frac{N}{k} \right\rceil$ is the ceiling value of $\frac{N}{k}$

Example: among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month.

Pigeonhole Principle

Example: how many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?

Solution: **4 Suits:** clubs ♣, Diamonds ♦, spades ♠, hearts ♥

We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\left\lceil \frac{N}{4} \right\rceil$ cards.

52: 13 ♣, 13 ♦, 13 ♠, 13 ♥

At least three cards of one suit are selected if $\left\lceil \frac{N}{4} \right\rceil \geq 3$. The smallest integer N such that

$$\left\lceil \frac{N}{4} \right\rceil \geq 3 \text{ is } N = 2 \times 4 + 1 = 9.$$

Pigeonhole Principle

Example: how many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 hearts are chosen? Note: Each suit has 13 cards.

Solution:

4 Suits: clubs ♣, Diamonds ♦, spades ♠, hearts ♥
52: 13 ♣, 13 ♦, 13 ♠, 13 ♥

A deck contains 13 hearts and 39 cards which are not hearts.

The worst case, we may selects all the clubs, diamonds, and spades (39 cards) before any hearts.

So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.

So, to guarantee that at least 3 hearts are selected, 39+3=42 cards should be selected.(generalized pigeonhole principle is not used here).