Basic Structures: 2.1 Sets

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Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics
  - Important for counting
  - Programming languages have set operations
- Set theory is an important branch of mathematics
  - Many different systems of axioms have been used to develop set theory

Sets

- A *set* is **an unordered collection** of objects
  - the students in this class
  - the chairs in this room
- The objects in a set are called the ***elements***, or ***members*** of the set. A set is said to ***contain*** its elements
- The notation $\boldsymbol{a} \in \boldsymbol{A}$ denotes that a is an element of the set $\boldsymbol{A}$
- If a is not a member of A, write $\boldsymbol{a} \notin \boldsymbol{A}$

Describing a Set: Roster Method

$S = \{a, b, c, d\}$

- Order not important

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, \ldots, z\}$$

Roster Method Examples

- Set of all vowels in the English alphabet:

  $V = \{a, e, i, o, u\}$

- Set of all odd positive integers less than 10:

  $O = \{1, 3, 5, 7, 9\}$

- Set of all positive integers less than 100:

  $S = \{1, 2, 3, \ldots, 99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3, -2, -1\}$

Some Important Sets

N = *natural numbers* = {0, 1, 2, 3, …}

Z = *integers* = {…, -3, -2, -1, 0, 1, 2, 3, …}

Z$^+$ = *positive integers* = {1, 2, 3, …}

R = set of *real numbers*

R$^+$ = set of *positive real numbers*

C = set of *complex numbers*

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy - Examples:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid P(x)\}$$

  - Example: $S = \{x \mid \text{Prime}(x)\}$
  - Example: Positive rational numbers

  $$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

(note '|' means 'such that')

Interval Notation

$$[a, b] = \{x \mid a \le x \le b\}$$
$$[a, b) = \{x \mid a \le x < b\}$$
$$(a, b] = \{x \mid a < x \le b\}$$
$$(a, b) = \{x \mid a < x < b\}$$

closed interval $[a, b]$

open interval (a, b)

Universal Set and Empty Set

- The *universal set* U is the set containing everything currently under consideration
  - Sometimes implicit
  - Sometimes explicitly stated
  - Contents depend on the context
- The empty set is the set with no elements. Symbolized $\emptyset$, but $\{\,\}$ also used.

Venn Diagram

Russell's Paradox

- Let S be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is S a member of itself?"

so that $S = \{x \mid x \notin x\}$???

S cannot be defined as this

- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Some Sets Characteristics

- Sets can be elements of sets.

  $\{\{1, 2, 3\}, a, \{b, c\}\}$

  $\{\mathrm{N}, \mathrm{Z}, \mathrm{Q}, \mathrm{R}\}$

- The empty set is different from a set containing the empty set

  $\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements

- If A and B are sets, then A and B are equal if and only if $\forall x\,(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets

$$\{1, 3, 5\} = \{3, 5, 1\}$$

$$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B
- $A \subseteq B$ holds if and only if $\forall x\,(x \in A \rightarrow x \in B)$ is true
  - $\emptyset \subseteq S$, for every set S
  - **Because** $a \notin \emptyset$ ($a \in \emptyset$ is always false)
  - $S \subseteq S$, for every set S
    - **Because** $a \in S \rightarrow a \in S$

Every element of set A is an element of set B

$\forall x(x \in \emptyset \rightarrow x \in S)$

Every element of set A is an element of set A

Showing a Set is or is not a Subset of Another Set

- To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B
- To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers

Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff $\forall x\,(x \in A \leftrightarrow x \in B)$
- Using logical equivalences we have $A = B$ iff
  $\forall x\,[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

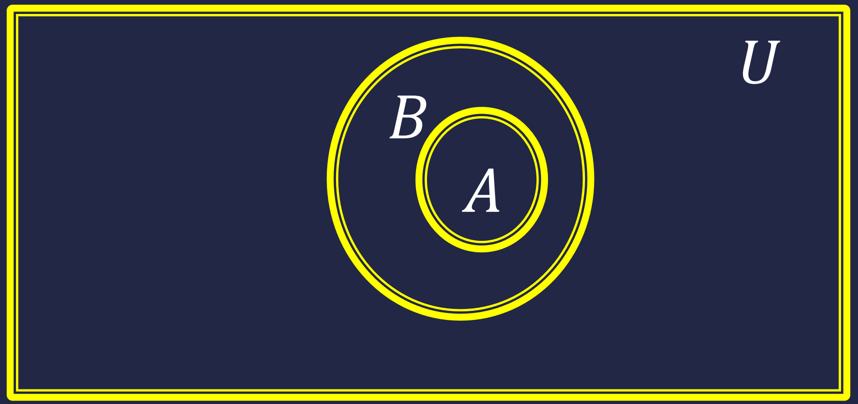
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a ***proper subset*** of B, denoted by $A \subset B$.

If $A \subset B$, then

$$\forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A) \text{ is true.}$$

Venn Diagram



The size of the set: Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1, 2, 3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the ***power set*** of A.

Example: If $A = \{a, b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n

Cartesian Products: Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots.., a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element
- Two ***ordered*** n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$

Cartesian Product

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

Example:

$A = \{a, b\}$ $B = \{1, 2, 3\}$

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a ***relation*** from the set A to the set B.

Cartesian Product

Definition: The Cartesian products of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

- $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \; i = 1, \ldots n\}$

Example: What is $A \times B \times C$ where $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$

Solution: $A \times B \times C =$ $\{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

Cartesian Product

- We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself.
- Similarly,
  - $A^3 = A \times A \times A$
  - $A^4 = A \times A \times A \times A$

Truth Sets of Quantifiers

- Given a predicate P and a domain D, we define the *truth set* of P to be the set of elements in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by

$$\{x \in D \mid P(x)\}$$

- **Example**: The truth set of $P(x)$ where the domain is the integers and $P(x)$ is "$|x| = 1$" is the set $\{-1, 1\}$

Where $|x|$ is the absolute value of x